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BY

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PREFACE

It is generally agreed among teachers of calculus that, while in a first course many students acquire some facility in manipulating calculus symbols, very few obtain a real understanding of the fundamental ideas of the subject. Thus, many students who could find the necessary derivatives and compute the curvature of a useless curve such as $y = e^{\text{arc sin } x^2}$ would fail miserably if asked to define in their own words either the derivative of a function or the curvature of a curve. Many students who are quite expert at formal differentiation and integration might be lost if asked to find the derivative of tan x with respect to sin x and explain the meaning of the result. And many students who can use Lhopital's rule when told to do so may fail to see its application if asked whether the quantity $2^x/x^{1000}$ is large or small when x is very large.

The difficulty is of course partly inherent in the subject matter itself. Certainly a large amount of purely formal work is necessary for a mastery of the technique. This technique is, however, of doubtful value unless accompanied by more understanding than we often find to be the case.

The primary object of the author in writing this book has been to present the subject with unusual clarity and simplicity without sacrificing accuracy. Every possible effort has been made to promote real understanding. Fundamental concepts and methods are presented in a simple and natural way; and an attempt is made to induce the student to view them as the natural and logical consequences of the thought that human beings have given to the type of problem involved. It is hoped that the book may be read with a greater than average amount of understanding and appreciation.

In order to secure a better understanding of the nature of the quantities considered, it has been found desirable to mention frequently the units in which they may be expressed. Thus, the derivative of the volume of a sphere with respect to its radius means much more to a beginner when he sees clearly that it might be expressed as so many cubic inches of increase in volume per inch of increase in radius. The curvature of a curve becomes much more real to the student as a rate of change of direction when he thinks of it in terms of a certain number of degrees or radians per unit of distance moved along the curve.

The first chapter provides a somewhat new approach to the subject. Here the student is taught to interpret the graph of $f(x)$ as a picture which shows how the value of $f(x)$ increases and decreases as x increases. Maximum and minimum values are defined. Many of the problems which appear later in connection with maxima, minima, and rates are first encountered in this introductory chapter. At this point the student is asked only to pick out the independent and dependent variables, set up the equation connecting them, sketch the graph, and answer certain questions by studying the graph. During the two years in which the book has been used in multigraphed form at Washington University, this preliminary instruction and practice in the troublesome problem of setting up the equation have proved to be exceedingly valuable. They have served also to stimulate interest and give the students a clearer understanding of the interpretation of the graph.

The differentiation of trigonometric functions is preceded by a very compact review of the definitions of these functions and the essential facts concerning them. Because many students have forgotten important trigonometric relations, this review serves a good purpose. It includes a set of problems and can be covered in a single assignment. A similar review of exponents and a brief discussion of the exponential and logarithmic functions precede the work on differentiation of these functions.

The author is indebted to his colleagues in the Department of Applied Mathematics, Washington University, for their cooperation in using the manuscript in multigraphed form, and to Mr. R. W. Bockhorst, instructor in engineering drawing, for making several of the isometric drawings.

ROSS R. MIDDLEMISS.

ST. LOUIS, MISSOURI,
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DIFFERENTIAL AND INTEGRAL CALCULUS

CHAPTER I

FUNCTIONS AND THEIR GRAPHS

1. Definition of a function. Notation.—Let x denote a variable to which values may be assigned arbitrarily; let y represent a variable that depends upon x in such a way that its value is determined when that of x is specified. We say, under these conditions, that *y is a function of x;* we call x the *independent variable* and y the *dependent variable.*

The student is familiar with an almost unlimited number of examples of such functional dependence of one variable quantity upon another. Thus the quantity $x^2 - 8x + 12$ is a function of x since its value is determined when that of x is specified; likewise the area of a circle is a function of its radius; the amount of one's monthly electric bill is a function of the amount of electrical energy used; the price of admission to a certain theater may depend upon the person's age.

The statement "y is a function of x" is abbreviated by writing

$$y = f(x),$$

which may be read "y equals f of x." Other letters may of course be used instead of f, and symbols such as $\varphi(x)$, $F(x)$, and $P(x)$ are frequently employed to denote different functions of x.

We use the symbol $f(a)$ to denote the value of $f(x)$ when x has the value a; thus, if

$$f(x) = x^2 - 8x + 12,$$

the value of $f(x)$ when $x = 1$ is

$$f(1) = 1^2 - 8(1) + 12 = 5.$$

Similarly,

$$f(2) = 2^2 - 8(2) + 12 = 0.$$
$$f(-4) = (-4)^2 - 8(-4) + 12 = 60.$$

2. Restrictions on the variables.—Our definition of a function, taken in its broadest sense, does not require the relation between x and y to yield a value of y for every possible value of x. Frequently x is restricted to a certain set of values, and it may happen that for some or all of these values, the relation yields more than one value of y. Thus, the relation

$$x^2 + y^2 = 16$$

defines y as a function of x only for $-4 \leqq x \leqq +4$ if we confine ourselves, as we shall in this work, to the field of real numbers. To each value of x in this interval, the relation yields two values of y. This is expressed by saying that y is a *double-valued* function of x.

In our work we shall deal almost exclusively with *single-valued* functions—functions that have a single definite value for each value of x under consideration. Multiple-valued functions are not, however, excluded, for such a function may be decomposed into two or more single-valued *branches* that may be considered separately. Thus, the branches of the above function are

$$y = \sqrt{16 - x^2} \qquad \text{and} \qquad y = -\sqrt{16 - x^2},$$

each of which is single-valued if we agree that the radical shall stand for the *positive square root* only.

3. Functional relations and analytical formulas.—The definition of a function does not imply that the relation between the variables can be expressed by an analytical formula or equation. It is only necessary that there be some means of finding the value of y when that of x is given.

Thus, the relation

$T(n)$ = the minimum official temperature in St. Louis on the nth day of the year 1938

satisfies all of the requirements. If n is given we can find T—not by substituting in a formula but by referring to the records of the Weather Bureau. Of course n is restricted to the set of integers from 1 to 365 inclusive. The function $T(n)$ is not defined by the above relation for n outside this set.

If the relation *can* be expressed analytically, it may happen that two or more equations are required—each equation expressing the relation for some definite set of values of the independent variable. Suppose, for example, that a cab driver charges 50 cents for 1 mile or less, and at a rate of 20 cents per mile for additional distance. The fare F (in cents) may be expressed analytically in terms of the distance d (in miles) by writing the two equations:

$$\begin{cases} F = 50 & 0 < d \leqq 1 \\ F = 30 + 20d & d > 1. \end{cases}$$

4. Graphical representation.—It is convenient to represent pictorially the relation between two variables by means of a graph. We assume that the student is familiar with the process of making a graph by constructing a table of corresponding values of the variables and plotting the points.

In his study of analytic geometry, the student learned to interpret the graph as the locus of all points whose coordinates satisfy the given relation. For our present purpose we must emphasize the following additional interpretation: The graph of $y = f(x)$ is a picture indicating the manner in which the value of y increases and decreases with increasing x. Specifically, the following points should be immediately clear:

1. *The value of the function is increasing* (as x increases) *in those intervals in which the curve is rising—and decreasing where the curve is falling.*

2. *The steepness of the curve is a general indication of the rapidity with which the value of the function is increasing or decreasing,* relative to x.

In order to fix these ideas let us examine carefully the graph of the function $A = \pi r^2$. It shows pictorially that the area of a circle increases continuously with r, starting at 0 when $r = 0$. The curve becomes continually steeper with increasing r. This means that the area increases more and more rapidly with r as r grows larger. This is evident from the following analysis, *which the student must understand clearly* before proceeding:

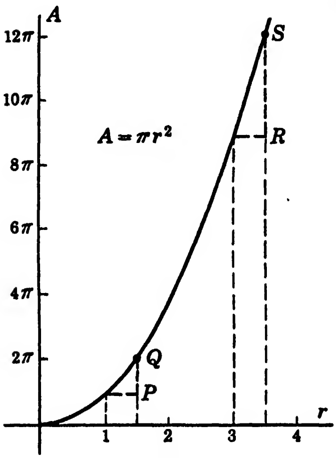

FIG. 1.

The area of a circle of 1 in. radius is represented by the ordinate to the curve at $r = 1$. If the radius is $1\frac{1}{2}$ in., the area is given by the ordinate at $r = 1\frac{1}{2}$. Consequently the length PQ represents the amount by which the area increases when the radius increases from 1 to $1\frac{1}{2}$ in. Similarly the length RS represents the increase in area when the radius increases from 3 to $3\frac{1}{2}$ in. Thus the *same* increase in r, starting at $r = 3$, produces a larger increase in A; *i.e.*, on the average, the area is increasing more rapidly with respect to r in this interval where the curve is steeper.

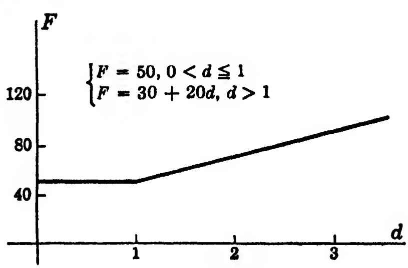

FIG. 2.

Figure 2 is the graph of the function

$$\begin{cases} F = 50 & 0 < d \leqq 1 \\ F = 30 + 20d & d > 1, \end{cases}$$

which was discussed in the preceding section. It shows

pictorially that the fare is *constant* for d from 0 to 1 and then increases *linearly* with d.

5. Increments.—Figure 3 is the graph of the function

$$y = \sqrt{x}.$$

At any point such as A, x and y have definite values, that of y being represented of course by the dotted ordinate. Suppose now that, starting at A, we let x increase by any small amount which we may represent by the symbol Δx. This symbol is read, **"increment of *x*,"** or "delta of x," or simply "delta x."

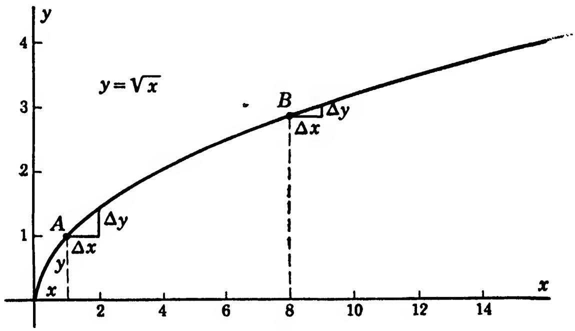

FIG. 3.

The change in the value of y, caused by this increment Δx in the value of x, is obviously represented by the length labeled Δy in the figure.

It is immediately clear that, since the curve continually rises, an increase in x starting at *any* point results in an *increase* in the value of y—*i.e.*, Δy is positive. It is furthermore obvious that the same increment in x produces *smaller and smaller* increments in y as we go farther out along this curve. Thus, as illustrated in the figure, $\sqrt{x}$ does not increase nearly so much when x changes from 8 to 9 as when it changes from 1 to 2.

The student should draw corresponding figures for such functions as x^2, $1/x$, $\log_{10} x$, etc., and study carefully the effect on the value of the function of a fixed small increase Δx in x, applied at different places along the curve. Note

that when a curve is falling, an increase in x results in a *decrease* in y—*i.e.*, Δy is negative if Δx is positive.

PROBLEMS

1. If $f(x) = x^2 - 5x + 4$, find $f(0)$, $f(1)$, $\cdots$, $f(5)$. Sketch the curve. What distances on the graph represent $f(0)$, $f(1)$, $f(3)$?

2. Sketch the curve $y^2 = x^3$. Does this equation define y as a single- or double-valued function of x if x is taken as the independent variable? What are the branches of the function? What is the situation if y is taken as the independent variable?

3. Express the surface area S of a cube as a function of its edge x. Sketch the function carefully. What length represents the surface area of a 1-in. cube? A $1\frac{1}{2}$-in. cube? What length represents the amount by which the surface area increases when the edge increases from 1 to $1\frac{1}{2}$ in.? From 3 to $3\frac{1}{2}$ in.?

4. Solve Prob. 3, using the volume instead of the surface area.

5. Illustrate graphically the fact that the *same* increment Δx applied to x, starting at different points on the line $y = ax + b$, produces the same change in y.

6. Sketch the curve $y = \sin x$ for x from 0 to 180°. What is the significance of the fact that the curve is steeper near $x = 10°$ than near $x = 80°$? How much does $\sin x$ increase when x increases from 10 to 11°? From 80 to 81°? Why does one not obtain $\sin 10\frac{1}{2}°$ exactly by the usual method of interpolation? Is the result too large or too small?

7. Sketch carefully on the same axes the curves $y = \sqrt{x}$, $y = x$, and $y = x^2$ for $x > 0$. Compare the manner in which these functions vary with x.

8. Sketch on the same axes the curves $y = x^2$, $y = x^4$, $y = x^{100}$. Compare the manner in which these functions vary with x as x increases from 0 to 2.

9. Sketch each of the following functions and discuss the manner in which it varies with x:

(*a*) $\log_{10} x$	(*b*) $\cos x$	(*c*) $\dfrac{1}{x}$
(*d*) 2^x	(*e*) 4^{-x}	(*f*) $\tan x$

10. A taxi company charges at the rate of 30 cents per mile but has a minimum charge of 45 cents. Express the fare analytically as a function of the distance and sketch the function.

11. At a certain theater the admission charge for adults is 60 cents. Children from one to twelve inclusive pay 20 cents, and those under one year of age are admitted free. Express the charge analytically as a

function of the age and sketch the function. Why is this called a *step-wise constant function?*

12. The rate for residence electric service in St. Louis County is as follows:

First 32 kw.-hr. per month at 5 cents per kw.-hr.
Next 168 kw.-hr. per month at $2\frac{1}{2}$ cents per kw.-hr.
All over 200 kw.-hr. per month at $1\frac{1}{2}$ cents per kw.-hr.
Minimum bill 50 cents.

Write equations expressing the bill as a function of the amount used. Sketch the function.

13. An unmarried person whose income for 1937 was not more than \$5,000 was required to pay a Federal income tax of 4 per cent on all over \$1,000 with the further provision that 10 per cent of income from wages was exempt from tax. Express analytically the tax as a function of the income, assuming that the income was derived entirely from wages. Sketch the function.

14. For n a positive integer let

$$f(n) = \text{the largest prime factor of } n.*$$

Does this relation satisfy the requirements of the definition of a function? Can you express it analytically? Construct the graph for $n = 1$ to $n = 12$.

15. For n a positive integer let

$\varphi(n) =$ the number of positive integers not greater than n and prime to n.†

Does this relation satisfy the requirements of the definition of a function? Construct the graph for $n = 1$ to $n = 15$.

6. Setting up the equation.—Most of the functions which we shall study are defined for all values of x in some interval, and the relation can be expressed by a single equation. In many of the applications of the calculus, the relation between the variables involved is stated in words—or is expressed by a combination of statements and drawings. Our first problem is to translate the information so given into an equation which expresses analytically the relation between the independent variable and the dependent variable or function. Having obtained the relation we

* A prime number is any integer that is different from 1 and has no divisor except itself and 1.

† Two integers are prime to each other if they have no common factor except 1. This function, which plays an important role in the theory of numbers, is called the *indicator* of n.

may study the manner in which the value of the function varies with that of the independent variable. The graph usually plays an important role. We shall illustrate the procedure by two examples which the student should study very carefully.

Example 1

From a 12- by 18-in. sheet of tin we wish to make a box by cutting a square from each corner and turning up the sides. How does the volume of the box obtained vary with the size of the square cut out? For about what size square would the largest box be obtained? See Fig. 4.

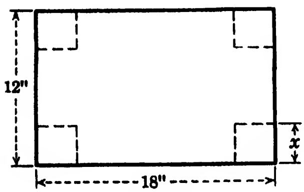

FIG. 4.

Solution

DEPENDENT VARIABLE—volume v of box.

INDEPENDENT VARIABLE—edge x of square.

RELATION: Since for any value of x the dimensions of the box are $18 - 2x$, $12 - 2x$, and x, we have

$$v = 4x(9 - x)(6 - x); \qquad 0 < x < 6. \quad \text{Why?}$$

DISCUSSION: The graphical picture of this relation between x and v is shown in Fig. 5. It shows that v increases from 0 to about 230 cu. in. as x increases from 0 to a little more than 2. Then v decreases as x increases and finally becomes 0 at $x = 6$. We shall learn later how to find exactly the value of x which gives the largest value of v.

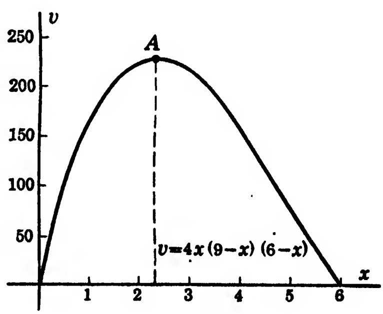

FIG. 5.

Example 2

A rectangular box to contain 108 cu. ft. is to be made with a square base. The cost per square foot of material for bottom, top, and sides is 1, 5, and 6 cents, respectively. Discuss the manner in which the cost varies with the dimensions.

Solution

DEPENDENT VARIABLE—cost C of material (dollars).

INDEPENDENT VARIABLE—we could choose either the edge of the base or the height. Take the former and denote it in feet by x.

RELATION: The height must be $108/x^2$. The area of the four sides is then $4 \cdot x \cdot 108/x^2 = 432/x$. The areas of top and bottom are each x^2. The cost is then

$$C = 0.01x^2 + 0.05x^2 + 0.06\frac{432}{x}$$
$$= 0.06\left(x^2 + \frac{432}{x}\right).$$

DISCUSSION: The graph (Fig. 6) shows that the cost would be very high if x were very small. Why? The cost decreases as x increases (rapidly at first and then more slowly) until x reaches the value 6, where it appears *perhaps* to be least. From this point on, the cost increases. The increase is slow at first, the cost being only slightly more for $x = 7$ or 8 than for $x = 6$.

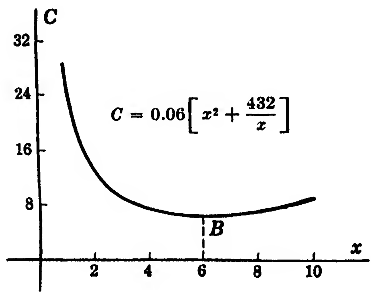

FIG. 6.

The examples just given make it clear that in setting up the equation we should proceed somewhat as follows:

1. *Pick out the dependent variable—the function whose properties are to be studied—and denote it by some letter.*

2. *Pick out the independent variable—the quantity that we may vary at will.* Often there is more than one possible choice for this variable.

3. *Obtain from the given statements and drawings the equation connecting the two variables.* This step requires a clear and deliberate analysis of the conditions of the problem.

7. Maximum and minimum values of a function.—The point A in Fig. 5 is called a *maximum point* on the curve; the corresponding value of the function is called a *maximum value.* The value of the independent variable x for which the maximum value is attained is sometimes called a *critical value* of x. Here the critical value of x is a little more than 2; the maximum value of the function appears to be about 230.

In general we say that a function $f(x)$ has a maximum value at $x = a$ if the value of $f(x)$ is larger when x *equals* a

than when x is *slightly more* or *slightly less* than a. A *minimum value* is similarly defined. The point B (Fig. 6) is a minimum point on the curve.

The following exercises are designed to give the student some practice in picking out the variables and expressing the relation between them in analytical form when the conditions of the problem are stated in words. They will serve also to impress upon him the following important fact:

Whatever the physical meaning of the variables may be, the relation between them, at least if it can be expressed analytically, can always be represented by a graph; the manner in which the value of the function varies with that of the independent variable may therefore be studied from the graph.

PROBLEMS

1. A radiator holds 14 qt. and is filled with a solution that is 30 per cent alcohol. Suppose we drain out x qt. and replace this by pure alcohol. Express the concentration of the resulting solution as a function of x. From the graph discuss the way in which the concentration varies with x.

2. A farmer has 100 yd. of fence and wishes to enclose a rectangular plot and divide it into two equal parts by a cross fence joining the mid-points of two sides. Express the area enclosed as a function of the width x. Sketch the function. What dimensions would appear to give the largest area?

3. From a log 8 in. in diameter and 12 ft. long a rectangular beam is to be sawed. Express the cross-sectional area of the beam as a function of its width. Make a graph and discuss the relation between the weight of the beam and its width.

4. With 80 yd. of woven wire it is desired to fence a rectangular plot of ground along the straight bank of a stream. No fence is necessary along the stream. Set up a function from which you can study the way in which the area enclosed varies with the dimensions of the field. Estimate the shape that would give the largest area.

5. A man in a rowboat at B, 6 miles from shore, wishes to reach a point A on the shore 10 miles away (Fig. 7). He can row 2 m.p.h. and walk 4 m.p.h. If he rows straight to A he will, of course, arrive in 5 hr. If he rows to the nearest point on shore and then walks 8 miles he will also arrive in 5 hr. Possibly the time would be less if he landed at some

intermediate point. Discuss this problem by selecting a proper independent variable, expressing the time in terms of it, and drawing a graph.

6. In a basement 8 ft. high it is desired to construct a coalbin against one wall. The capacity is to be 240 cu. ft. Express the amount of sheathing required as a function of the dimensions; make a graph and determine for what dimensions the amount of lumber appears to be least.

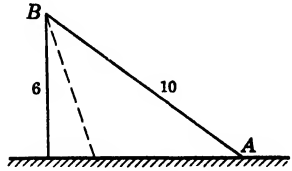

FIG. 7.

7. Solve Prob. 6 for the case in which the bin is built in one corner of the basement.

8. A ladder 16 ft. long stands vertically against a wall. The lower end is then pulled away from the wall at a uniform rate. Express the distance y of the upper end above the floor as a function of the distance x of the lower end from the wall. Sketch the function. What length on the graph represents the amount by which the top slides down when the bottom is pulled out the first foot? The fourth foot? The last foot? Does the top slide down the wall at a uniform speed, or in what general way does its speed vary?

9. A boy 5 ft. tall walks directly away from a lamppost 15 ft. high on which there is a light. Express the length S of his shadow as a function of his distance x from the post. Sketch this function. If he walks at a constant speed does his shadow lengthen also at a constant rate?

10. A ship B is 40 miles due east of a ship A. B starts north at 25 m.p.h., and at the same time A starts in the direction north $\theta°$ east, where $\theta = \text{arc tan } \frac{4}{3}$, at 15 m.p.h. Express the distance D between them at the end of T hours as a function of T. From a graph try to determine approximately when they will be closest together. HINT: It will be easier to plot D^2 instead of D, against T.

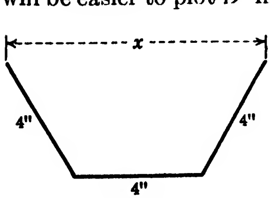

FIG. 8.

11. A gutter is to be made from a long piece of sheet metal that is 12 in. wide by folding up one-third of the width on each side as indicated in Fig. 8. Express the area of the cross section as a function of the width x across the top. Sketch the function for $x = 4$ in. to $x = 12$ in. What width appears to give the greatest capacity?

12. Water is being poured into a right circular cone that is 10 in. across the top and 10 in. deep. Express the volume of water in the cone at any instant as a function of its depth y.

13. A man operates a rooming house containing 24 rooms. He estimates that he could keep all of the rooms rented at \$16 per month each but would have one vacancy for each dollar per month added to this

price; he estimates also that he saves \$2 per month in maintenance expense on each vacant room. Set up a function from which you can study the way in which his net income varies with the rental. Try to determine from a graph the rental that would yield the greatest net income.

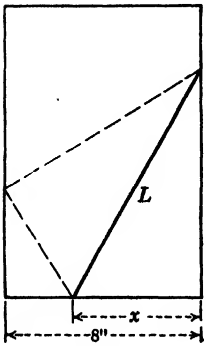

FIG. 9.

14. One corner of a sheet 8 in. wide is folded back so as to touch the opposite edge of the sheet. Express the length L of the crease as a function of x (Fig. 9) and study the way in which it varies with x.

15. The radius and height of a right circular cone (Fig. 10a) are $r = 6$ in. and $h = 8$ in. A right circular cylinder of radius x $(0 < x < 6)$ is inscribed in the cone. Express the volume of the cylinder as a function of x and make a graph. What value of x appears to give the largest cylinder?

16. Solve the above problem, considering the lateral area of the cylinder instead of its volume.

17. A right circular cylinder is inscribed in a sphere of radius r. Express its volume as a function of its height h. Sketch the function.

18. A right circular cone is circumscribed about a sphere of radius 6 in. Express its volume as a function of its height h. Sketch the function

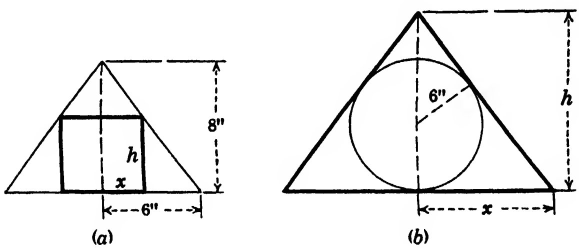

FIG. 10.

carefully. Does it appear to have a minimum value? HINT: Note that $V = \frac{1}{3}\pi x^2 h$ and that $\frac{x}{h} = \frac{6}{\sqrt{(h-6)^2 - 6^2}}$ (Fig. 10b).

19. Explain the statement "$F(a)$ is a maximum value of $F(x)$ if $F(a) > F(a \pm \epsilon)$ where ϵ is a small positive constant." Illustrate with a figure.

CHAPTER II

THE LIMIT OF A FUNCTION

8. Introduction.—The fundamental problem of the differential calculus is that of determining the rate at which the value of a function $f(x)$ changes, relative to x.

Suppose, for example, that one is inflating a toy spherical balloon. At a given instant the radius has a certain value r and the volume V is $\frac{4}{3}\pi r^3$; since the radius is increasing at this instant, the volume is also increasing. We may ask, "At what rate is the volume increasing, *relative to the radius*—*i.e.*, the volume is increasing at a rate of how many cubic inches per inch of increase in the radius?" We may, of course, ask a similar question concerning the rate at which the surface area is increasing.

Before attempting to answer such questions we must digress in the present chapter in order to study the important idea of the limit of a function—for it is upon this idea that the answer is based. The student has already employed the conception of a limit in several instances. The perimeter of a circle is defined, for example, as the *limit* of the perimeter of the inscribed or circumscribed regular polygon of n sides as n increases indefinitely and the length of each side approaches zero.

9. The limit of $f(x)$ as x approaches a constant.—We state that the limit of a certain function $f(x)$ as x approaches some constant a is the number L, by writing

$$\lim_{x \to a} f(x) = L.$$

The statement means, speaking roughly at first, that the value of $f(x)$ can be made to come as near to L as we please if we ~~take x~~ sufficiently near to a, but not equal to a. It

has nothing whatever to do with the value of $f(x)$ when *x equals a.*

As an example, we may write

$$\lim_{x \to 3} (x^2 - 1) = 8;$$

for obviously the value of $x^2 - 1$ is *arbitrarily near* 8 if x is *sufficiently near* 3. It happens also that the value of this function *is* 8 when x *equals* 3, but that fact has nothing to do with the present idea.

The above example will perhaps appear trivial. To cite an example that is not so obvious, we may state, as will be proved later, that

$$\lim_{x \to 0} \frac{3 \sin 4x}{2x} = 6.$$

This function has no value at all if x equals 0; for substituting 0 for x leads to the meaningless symbol 0/0.* For all values of x *near* 0 however, the value is *near* 6. In fact, if we choose any small positive number ϵ (such as 0.00001), the difference between the value of this function and 6 is less than this ϵ (in absolute value), for *all* values of x sufficiently close to 0, but $\neq 0$; *i.e.*, for all values of $x \neq 0$ which differ from 0 by less than some corresponding small amount δ.

This last statement expresses more precisely what is meant by saying that

$$\lim_{x \to a} f(x) = L.$$

It means that, after choosing *any* positive number ϵ, however small, another positive number δ can be found such that

$$|L - f(x)| < \epsilon$$

* A function is said to be *undefined* for any value of x for which the given relation does not yield a value of the function. The symbol 0/0 is discussed in the next article.

for every value of x satisfying the inequality

$$0 < |a - x| < \delta.$$

A value of δ corresponding to a given ϵ may sometimes be found easily as is indicated in the first problem of the next set.

10. The symbols 0/0 and a/0.—In explanation of the statement which was made in the last section about the symbol 0/0, it is well to examine carefully the definition of the symbol N/D in general.

It will be recalled that if N is any number, and D is any number *except* 0, there exists a *uniquely determined number Q such that*

$$Q \cdot D = N.$$

The symbol N/D is *defined* to stand for this number Q. Thus,

$$\tfrac{16}{2} = 8 \qquad \text{and} \qquad \tfrac{0}{6} = 0.$$

It is clear that this definition assigns no meaning whatever to the symbol 6/0; there is, in fact, no number Q such that $Q \cdot 0 = 6$. The symbol 6/0 has then, as yet, no more meaning than $\square/\Delta$; it does not represent any number. If a meaning is to be assigned to it, such meaning must be obtained by some extension of the original definition.

Likewise the symbol 0/0 is left meaningless by the above definition of N/D. We do not choose here to assign any value or meaning to these symbols.

11. Continuity.—Figure 11 is the graph of the function

$$f(x) = \sqrt{x}.$$

Let us examine carefully the part of the curve in the neighborhood of some particular point—say near $x = 4$. It is evident that

1. When x *equals* 4, the value of $f(x)$, represented by the solid ordinate, is 2.

2. When x has any value very *near* 4, the value of $f(x)$, represented by dotted ordinates, is very *near* 2. In fact if

x is sufficiently near 4, the value of $f(x)$ is *arbitrarily* near 2; *i.e.*,

$$\lim_{x \to 4} f(x) = 2.$$

Under these conditions $f(x)$ is said to be *continuous* at the point where $x = 4$. In general, a function $f(x)$ is said to be

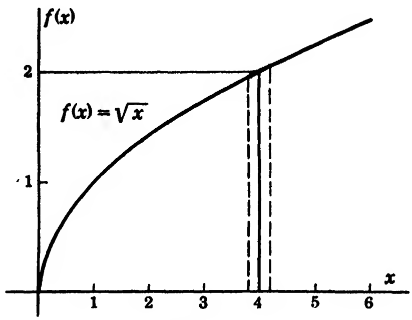

FIG. 11.

continuous at a point where $x = a$ if the value of $f(x)$ when x equals a is identical with the *limit* of $f(x)$ as x approaches a. If this condition is satisfied at every point of a certain interval, the function is said to be *continuous over the interval;* in this case its graph in this interval is a continuous curve in the ordinary sense of the word.

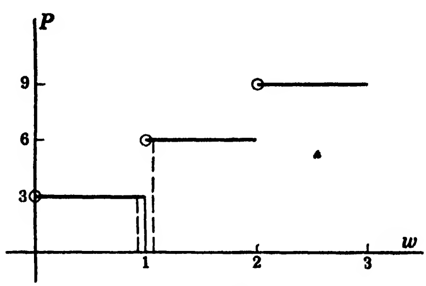

FIG. 12.

As an example in which the condition is *not* satisfied, consider Fig. 12, which is the graph of the postage P required on a letter as a function of its weight w (in ounces). The postage is 3 cents for each ounce or *fraction thereof.*

g the part in the neighborhood of $w = 1$ we see

en w *equals* 1, the value of the function P is 3, *i.e.*,
age is 3 cents if the weight is exactly 1 oz.
is not necessarily true that if w is *near* 1, P is *near* 3;
if w exceeds 1 by any amount δ however small,
Hence,

$$\lim_{w \to 1} P \text{ does not exist.}$$

The function is then discontinuous at this point. The small circle drawn around the end point of a segment indicates that this point is *not* a part of the graph.

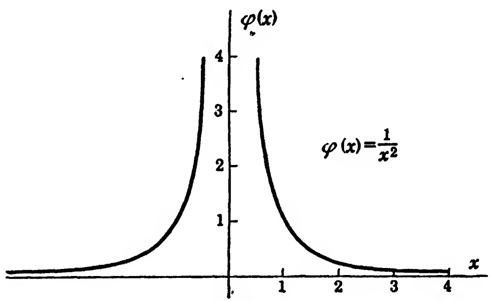

FIG. 13.

Another type of discontinuity is illustrated by Fig. 13, which is the graph of

$$\varphi(x) = \frac{1}{x^2}.$$

Examining the function in the neighborhood of $x = 0$ we find that

1. When x *equals* 0 the function has no value at all; it is *undefined.*

2. When x is *near* 0 the value of $\varphi(x)$ is very large; in fact it is *arbitrarily large* for x sufficiently near 0. Certainly there is no constant L to which the value of $\varphi(x)$ is *near* if x is near 0; *i.e.*,

$$\lim_{x \to 0} \frac{1}{x^2} \text{ does not exist;}$$

hence the function is discontinuous at $x = 0$.

The behavior of this function in the neighbor $x = 0$ is described by saying that as x approa the value of the function "increases without lin "increases beyond bound," or "approaches infinity, "becomes infinite." Symbolically, this is abbreviated writing

$$\lim_{x \to 0} \frac{1}{x^2} = \infty, \qquad \text{or} \qquad \frac{1}{x^2} \to \infty \text{ as } x \to 0.$$

The student should see clearly that we do not here assign any meaning to the word "infinity" when standing alone. We define only the phrase "approaches infinity." It is important to notice also that we do *not* say that $1/0$ *equals* infinity; in fact we have said nothing whatever about the value of $1/x^2$ when $x = 0$. We have merely coined the phrase "approaches infinity" to describe the obvious fact that the value of $1/x^2$ is *arbitrarily large* if x is sufficiently *near* 0.

12. The limit of $f(x)$ as x increases indefinitely.—The concept of the limit of $f(x)$ as x approaches a is one which concerns the behavior of $f(x)$ when x is near a. An equally important conception is that which concerns the behavior of $f(x)$ when x increases indefinitely.

It may happen that, as x becomes larger and larger, the value of a function $f(x)$ approaches closer and closer to a fixed constant L; more precisely, it may happen that the value of $f(x)$ is *arbitrarily near* L for all sufficiently large values of x. We say then that L is the limit of $f(x)$ as x increases indefinitely, and write

$$\lim_{x \to \infty} f(x) = L.$$

Thus it is obvious that the values of the function $\left(2 + \frac{1}{x}\right)$ are arbitrarily near 2 for all sufficiently large values of x;

write

$$\lim_{x \to \infty} \left(2 + \frac{1}{x}\right) = 2.$$

use the same notation to indicate that the value of omes indefinitely large if that of x does. Thus we rite

$$\lim_{x \to \infty} (x^2 - 1) = \infty.$$

PROBLEMS

1. Is it obvious that the value of $x^2 + 1$ is arbitrarily near 5 if x is sufficiently near 2? How close must x remain to 2 in order that the difference between $x^2 + 1$ and 5 shall be less than 0.1? Illustrate graphically.

HINT: If $x^2 + 1 < 5.1$, $x^2 < 4.1$ and $x < \sqrt{4.1}$. If $x^2 + 1 > 4.9$, $x^2 > 3.9$ and $x > \sqrt{3.9}$.

2. Is it obvious that the value of the function $3 + \frac{5}{x^2}$ is arbitrarily near 3 for all sufficiently large values of x? How large must x be in order that the difference may be less than 0.01? Illustrate graphically.

3. Is the area of a circle a continuous function of its radius? Is the parcel post charge on a 5-lb. package a continuous function of the distance to its destination? Illustrate graphically.

4. If x is a variable that takes successively the values $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \cdots$, we say that it is approaching 2 as a limit. In what sense are we here speaking of the limit of a *function?* Can one ever speak properly of the limit of an "isolated" variable?

5. Express the perimeter of the regular polygon of n sides inscribed in a circle of radius r as a function of n. What does the limit of this function represent as n increases indefinitely?

6. Solve Prob. 5 for the area of the circumscribed polygon.

7. Sketch the function $y = 2 + \frac{1}{x}$ and show the graphical meaning of the fact that the limit of this function, as x increases indefinitely, is 2.

8. Sketch the function $y = \frac{3x - 1}{x + 1}$. What is the value of $\lim_{x \to \infty} \frac{3x - 1}{x + 1}$?

9. Explain the following statement, illustrating with a graph: $f(x)$ is continuous at $x = a$ if, after choosing a positive number ϵ, how-

ever small, a positive number δ can be found such that

$$|f(x) - f(a)| < \epsilon \text{ if } |x - a| < \delta.$$

10. Draw sketches to show that in general the δ of Prob. on both ϵ and x; *i.e.*, with a fixed ϵ the value of δ changes as along the curve, and at a fixed point on the curve δ chang change ϵ.

13. Determination of the limit. Theorems on limi We consider now the problem of determining the limit function $f(x)$ as $x \to a$, or as $x \to \infty$.

In many cases the problem is trivial, the limit bei immediately apparent. Thus, we find that

$$\lim_{x \to 3} (x^3 - 3x + 2) = 20$$

by observing that if x is near 3, x^3 is near 27, $3x$ is near 9, and hence $(x^3 - 3x + 2)$ must be near 20. It may also be observed that this function is everywhere *continuous*—and that its limit as x approaches any constant a is merely the value which it has when x *equals* a; it may therefore be found by direct substitution.

In some cases in which the limit is not apparent, it can be made apparent by writing the function in a different form. Thus,

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

is not immediately evident. If x is near 4, both numerator and denominator are near 0; the behavior of the value of the function when $x \to 4$ is not at first apparent. However, if we write the function in the form

$$\left(\frac{x - 4}{x - 4}\right)(x + 4),$$

we see immediately that the first factor is *exactly* 1 for all values of x near 4 (but $\neq 4$), while the second is arbitrarily near 8 for x sufficiently near 4. Hence

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = 8.$$

The same result is obtained by noting that

$$\frac{x^2 - 16}{x - 4} \equiv x + 4 \text{ if } x \neq 4.$$

Since the value when x *equals* 4 is not involved in finding the limit in question we may say that

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4),$$

and this latter limit is obviously 8.

In these examples we have used two of the following theorems: If

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M,$$

then

(1) $$\lim_{x \to a} [f(x) + g(x)] = L + M.$$

(2) $$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M.$$

(3) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ if } M \neq 0.$$

Stated informally, the proof of the first theorem is as follows: By hypothesis, for x sufficiently near a,

$f(x)$ is arbitrarily near L;
$g(x)$ is arbitrarily near M.

Hence,

$f(x) + g(x)$ is arbitrarily near $L + M$.

But this last statement means that

$$\lim_{x \to a} [f(x) + g(x)] = L + M.$$

When the student has grasped the simple idea involved in this informal proof, he can easily write out a more formal proof using the ϵ and δ notation employed in the formal definition of the limit of a function. The proofs of the

second and third theorems are of course similar to that for the first.

One sometimes expresses the fact that the limit of a certain function $f(x)$ as x approaches a is L by saying, somewhat carelessly, "$f(x)$ approaches L as x approaches a," and writing

$$f(x) \to L \qquad \text{as} \qquad x \to a.$$

This notation is open to criticism on the ground that it does not adequately convey the notion of the *limit* of a function; it is sometimes convenient, however, and when used it should be understood to mean exactly the same thing as the previous notation.

14. The limit of $\dfrac{\sin x}{x}$ as x approaches 0.—This limit, which is of great importance in our later work, cannot be obtained by inspection. If x is near 0, both numerator and denominator are near 0 as in the last example—but in this case we cannot easily transform the function so as to make the limit evident. In order to find it we shall borrow some simple facts from geometry.

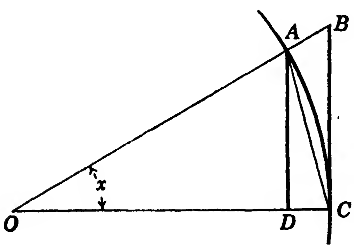

FIG. 14.

Represent x (in radians) as a central angle in a circle of radius $OC = 1$ (Fig. 14). Then

$$\begin{aligned} \text{Area of } \triangle OBC &= \tfrac{1}{2} OC \cdot BC &&= \tfrac{1}{2} \tan x. && \text{Why?} \\ \text{Area of sector } OAC &= \tfrac{1}{2} \overline{OC}^2 \cdot x &&= \tfrac{1}{2} x. && \text{Why?} \\ \text{Area of } \triangle OAC &= \tfrac{1}{2} OC \cdot AD &&= \tfrac{1}{2} \sin x. && \text{Why?} \end{aligned}$$

Of these three areas the first is largest and the last is smallest; *i.e.*,

$$\tfrac{1}{2} \tan x > \tfrac{1}{2} x > \tfrac{1}{2} \sin x.$$

Dividing by $\frac{1}{2} \sin x$ this becomes

$$\frac{1}{\cos x} > \frac{x}{\sin x} > 1.$$

Inverting (and reversing the inequality sign) we have

$$\cos x < \frac{\sin x}{x} < 1.$$

This inequality states that the value of $\dfrac{\sin x}{x}$ lies between that of $\cos x$ and 1. But the limit of $\cos x$ as $x \to 0$ is 1. Hence $\dfrac{\sin x}{x}$, since it lies between 1 and a quantity which is approaching 1 as a limit, must also approach 1 when $x \to 0$. That is

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Using this fundamental limit and the theorems of the last section, one can solve many limit problems in which trigonometric functions are involved.

Example 1

Find $\displaystyle\lim_{x\to 0} \frac{3 \sin 4x}{2x}$.

Solution

$$\begin{aligned}\lim_{x\to 0} \frac{3\sin 4x}{2x} &= \lim_{x\to 0}\left(6 \cdot \frac{\sin 4x}{4x}\right)\\ &= \lim_{x\to 0} 6 \cdot \lim_{x\to 0} \frac{\sin 4x}{4x}\\ &= 6 \cdot 1 = 6.\end{aligned}$$

Example 2

Find $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2}$.

Solution

$$\begin{aligned}\lim_{x\to 0} \frac{1 - \cos x}{x^2} &= \lim_{x\to 0} \frac{2 \sin^2 \frac{1}{2}x}{x^2}\\ &= \lim_{x\to 0}\left(\tfrac{1}{2} \cdot \frac{\sin \frac{1}{2}x}{\frac{1}{2}x} \cdot \frac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right)\\ &= \tfrac{1}{2} \cdot 1 \cdot 1 = \tfrac{1}{2}.\end{aligned}$$

PROBLEMS

Compute the following limits if they exist:

1. $\displaystyle\lim_{x\to 2} (x^2 - x + 4)$.
2. $\displaystyle\lim_{x\to 4} \frac{3x + 4}{x - 2}$.

3. $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$.

4. $\lim\limits_{x \to 3} \dfrac{x^4 - 81}{x^2 - 9}$.

5. $\lim\limits_{x \to 4} \dfrac{x^2 - 2x - 8}{x^2 + 2x - 24}$.

6. $\lim\limits_{x \to 2} \dfrac{x^2 + 6x}{x - 2}$.

7. Show that $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$. HINT: $\tan x = \dfrac{\sin x}{\cos x}$.

8. Show that the value of $\dfrac{4 - 4 \cos x}{x^2}$ is near 2 if x is near 0.

9. Show that the value of $\dfrac{x^2 - x - 6}{x^2 - 7x + 12}$ is near -5 if x is near 3.

10. Evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x \sin x}$.

11. What can you say about $\lim\limits_{x \to \infty} f(x)$ if $f(x)$ is a polynomial in x?

12. Find $\lim\limits_{x \to \infty} \dfrac{3x^2 + 5x - 7}{x^2 + 2x + 4}$. HINT: Divide each term by x^2. Note also that for large values of x, the function has the behavior of $3x^2/x^2$. Why?

13. Let $f(x) = p_m(x)/p_n(x)$ where $p_m(x)$ and $p_n(x)$ are polynomials in x of degree m and n respectively. What can you say about $\lim\limits_{x \to \infty} f(x)$ if $m > n$, $m = n$, and $m < n$?

14. Express the perimeter of the regular polygon of n sides inscribed in a circle of radius r as a function of n. Compute the limit of this function as $n \to \infty$. Is this an acceptable proof of the formula $C = 2\pi r$, or has the formula been assumed?

15. Solve Prob. 14 for the area of the inscribed polygon.

16. The function $f(x) = \dfrac{3x + 4 \tan x}{x}$ is undefined at $x = 0$. What value must be assigned to $f(0)$ if $f(x)$ is to be continuous at this point?

The sum of a geometric progression is $S = a\dfrac{1 - r^n}{1 - r}$ where a is the first term, r the common ratio, and n the number of terms. In each of the following, express the sum S_n of n terms as a function of n; then compute $\lim\limits_{n \to \infty} S_n$. This limit is, by definition, the "sum" of the "infinite progression."

17. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

18. $9 + 3 + 1 + \frac{1}{3} + \cdots$.

19. $0.636363 \cdots$.

20. For what range of values of r does $\lim\limits_{n \to \infty} S_n$ exist, and what is its value?

CHAPTER III

THE DERIVATIVE

15. Definition.—The manner in which the value of a function $f(x)$ varies with x was studied in a general way in Chap. I. We saw there that if the graph of the relation

$$y = f(x)$$

has the form shown in Fig. 15, the value of y is increasing, as x increases, throughout an interval such as AB, where the curve is rising.

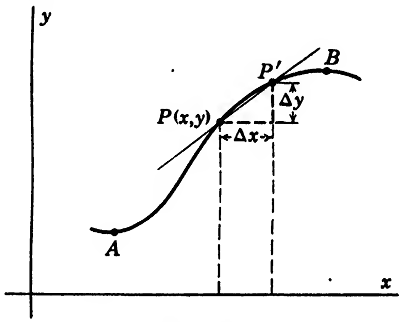

FIG. 15.

Let us now select any point $P(x, y)$ of this interval and ask, "At what rate is y increasing with respect to x at this point—*i.e.*, y is increasing how many times as fast as x, or at a rate of how many units per unit of increase in x?"

A little reflection will convince the student that in order to answer this question we should proceed as follows: Starting at P,

1. Let x increase by a small amount Δx, and determine the amount by which y increases.* Call this Δy.

* The student should draw the corresponding figure for the case in which the value of the function is decreasing—noting in this case that an increase in x produces a *decrease* in y; *i.e.*, Δy is negative.

2. Divide the increase in y by that in x. This quotient $\Delta y/\Delta x$ is called the *average* rate of increase of y with respect to x over this interval. Why?

3. Now, think of taking for Δx smaller and smaller values. The corresponding set of values of $\Delta y/\Delta x$ represents the average rate of change of y with respect to x over *smaller and smaller intervals beginning at P.* Clearly we should define the *instantaneous* rate of change *at P* as the limit of the quotient $\Delta y/\Delta x$ as $\Delta x \to 0$.

This limit is called the *derivative of the function with respect to x.* It is denoted by any of the symbols,

$$\frac{dy}{dx}; \qquad f'(x); \qquad y'; \qquad D_x y.$$

Its value at any point (x_1, y_1) is denoted by

$$\left.\frac{dy}{dx}\right]_{x_1, y_1}; \qquad f'(x_1); \qquad y']_{x_1, y_1}; \qquad D_x y]_{x_1, y_1}.$$

Since Δy is obviously equal to $f(x + \Delta x) - f(x)$, we may write the definition of the derivative symbolically as follows:

$$\frac{dy}{dx} \text{ or } f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.^{*}$$

16. Graphical meaning of the derivative.—Referring to Fig. 15, we see that $\Delta y/\Delta x$ is the slope of the secant line PP'. Obviously, as $\Delta x \to 0$ and P' approaches P along the curve, the secant line turns about point P approaching a position of tangency to the curve at P. Hence, *the value of the derivative at P is the slope of the tangent to the curve at this point.*

17. Differentiation.—We consider now the problem of finding the derivative of a given function. From the definition it is evident that the process, which is called *differentiation,* requires the following three steps:

1. Obtain Δy (in terms of x and Δx) by subtracting $f(x)$ from $f(x + \Delta x)$;

$$\Delta y = f(x + \Delta x) - f(x).$$

* $f(x)$ is said to be *differentiable* at P if this limit exists. All functions used here will be differentiable except possibly at some points.

2. Divide Δy by Δx;

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

3. Determine the limit of this quotient as $\Delta x \to 0$. This limit is dy/dx or $f'(x)$.

The following examples illustrate both the process of finding the derivative and its physical and geometrical interpretation. The student should therefore study them carefully.

Example **1**

For the function $y = x^2$ find the value of dy/dx at any point. What is its value at the point (2, 4)?

Solution (Fig. 16)

Starting at any point $P(x, y)$ and letting x increase by a small amount Δx we have

(1) $$\Delta y = (x + \Delta x)^2 - x^2 = 2x\,\Delta x + \overline{\Delta x}^2.$$

(2) $$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

(3) $$\frac{dy}{dx} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

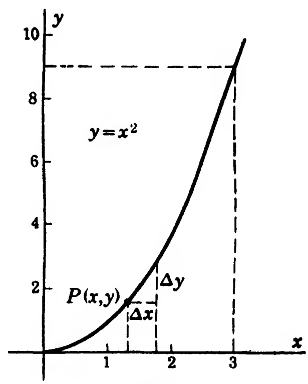

FIG. 16.

The value of dy/dx at *any* point (x, y) is $2x$. The value at (2, 4) is therefore 4. This means that the slope of the tangent to the curve at this point is 4. It also means that y is increasing 4 times as fast as x when we go through this point in tracing the curve.

Example **2**

Given $f(x) = 3x + \frac{1}{x}$, find $f'(x)$.

Solution

(1) $$\Delta f(x) = \left[3(x + \Delta x) + \frac{1}{x + \Delta x}\right] - \left(3x + \frac{1}{x}\right)$$

$$= \frac{3x^2\Delta x + 3x\,\overline{\Delta x}^2 - \Delta x}{x(x + \Delta x)}.$$

(2) $$\frac{\Delta f(x)}{\Delta x} = \frac{3x^2 + 3x\,\Delta x - 1}{x(x + \Delta x)}.$$

(3) $$f'(x) = \lim_{\Delta x \to 0} \frac{3x^2 + 3x\,\Delta x - 1}{x(x + \Delta x)} = \frac{3x^2 - 1}{x^2}.$$

Example 3

Suppose that the edge x of a cube is increasing; find the rate at which the volume is increasing with respect to x when $x = 2$ in.

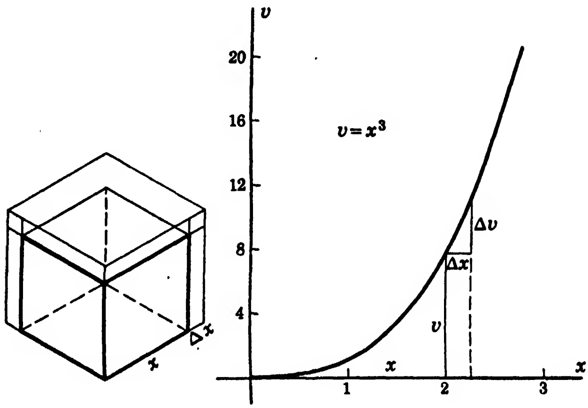

FIG. 17.

Solution (Fig. 17)

When the edge has an' 'ength x(inches) the volume is

$$v = x^3 \qquad \text{(cu. in.)}.$$

If now the edge increases by a small amount Δx, the increase in volume is

(1) $$\Delta v = (x + \Delta x)^3 - x^3$$
$$= 3x^2\Delta x + 3x\,\overline{\Delta x}^2 + \overline{\Delta x}^3.$$

(2) $$\frac{\Delta v}{\Delta x} = 3x^2 + 3x\,\Delta x + \overline{\Delta x}^2.$$

(3) $$\frac{dv}{dx} = \lim_{\Delta x \to 0} (3x^2 + 3x\,\Delta x + \overline{\Delta x}^2) = 3x^2.$$

$$\left.\frac{dv}{dx}\right]_{x=2} = 12.$$

This means that at this instant the volume is increasing 12 times as rapidly as x—or at a rate of 12 *cu. in. per inch of increase in* x. If the edge were increasing at say 3 in. per minute, the volume would of course be increasing at $12 \cdot 3 = 36$ *cu. in. per minute.*

It should be noticed that we always reduce (1) *to a form containing* Δx *as a factor* before writing (2). The second step then merely cancels this factor and the third step is usually obvious. Certain cases in which this cannot be done will be considered later, but for the present the student should always follow this procedure. Why may we cancel the factor Δx from numerator and denominator before seeking the limit of the fraction as $\Delta x \to 0$?

PROBLEMS

1. Show that at any point in an interval in which $f(x)$ is decreasing (as x increases), the value of dy/dx is negative. (See footnote, p. 25.)

2. Is it necessary that the limit of Δy (regarded as a function of Δx) shall be 0 as $\Delta x \to 0$ in order that the limit of $\Delta y/\Delta x$ may exist? Does this require that the function be continuous at the point under consideration?

In each of the following, first sketch the curve and select any point $P(x, y)$ on it; then compute the derivative at P. Show the increments.

3. $y = x^2 - 4$.

4. $y = x^2 - 7x + 10$.

5. $y = 2x^2 - 5x - 12$.

6. $\varphi(t) = 4t^2 - 8t$.

7. $f(x) = ax + b$.

8. $y = ax^2 + bx + c$.

9. $y = \dfrac{1}{x}$.

10. $w = \dfrac{1}{x^2}$.

11. $y = \dfrac{1}{x^2 - 1}$.

12. $y = \dfrac{6}{x^2 + 1}$.

13. $y = \dfrac{x - 3}{x + 3}$.

14. $y = \dfrac{x}{x - 1}$.

15. $y = x^3$.

16. $y = x^3 - 16x$.

17. $w = (t - 2)^3$.

18. $y = x^3 - 3x^2 - 6x + 8$.

19. $y = x^4$.

20. $y = x^3 + \dfrac{1}{x^2}$.

Compute the slope of the tangent line to each of the following curves at the given point. Sketch the curve and tangent line.

21. $y = x^2 + 2x - 6$, $(2, 2)$.

22. $y = x^2 - 6x + 5$, $(3, -4)$.

23. $\varphi(x) = \dfrac{9}{x}$, $(-3, -3)$.

24. $y = x^3 - 2x$, $(1, -1)$.

25. $y = x^3 - 6x^2 + 3x + 10$, $(2, 0)$.

26. $y = \dfrac{8}{x^2 + 4}$, $(0, 2)$.

27. Sketch the curve $y = x^2 - 4$. Suppose that a point moves along this curve from left to right so that its abscissa increases uniformly at 3 units per minute. At what rate is its ordinate increasing when it goes through $(1, -3)$? HINT: You will find that $\left.\frac{dy}{dx}\right]_{1,-3} = 2$; that is, y is increasing twice as fast as x.

28. In the preceding problem compute the time rate of change of the ordinate when the moving point passes through $(3, 5)$. Through $(-1, -3)$. Why is the last result negative?

29. Sketch the parabola $y = 5 + 4x - x^2$. Obtain an expression for the slope of the tangent at any point. Find the slope at the points where the curve crosses the axes. Find the point where the slope is zero.

30. Sketch the curve $V = \frac{4}{3}\pi r^3$. Find the rate of change of the volume with respect to the radius (in cubic inches per inch) when $r = 2$ in. If you were inflating a toy spherical balloon and the radius were increasing at $\frac{1}{2}$ in. per minute, at what rate would the volume be increasing at the instant when $r = 2$ in.?

31. In the above problem we have spoken of the rate of increase of the volume of a sphere *with respect to its radius.* Is it obvious to you that this is equal to the rate at which the volume is increasing with respect to the time (in say cubic inches per minute) divided by the rate at which the radius is increasing (in inches per minute)?

32. Find the rate of change of the surface area of a cube with respect to its edge x. In what units is this expressed if x is in inches? If x is 4 in. and is increasing at 2 in. per minute, at what time rate is the surface area increasing?

33. Find the rate of change of the volume of a cube with respect to its edge (in cubic inches per inch). Find also the rate of change of the surface area with respect to the edge (in square inches per inch). If the first of these is divided by the second, what does the quotient represent and in what units is it expressed? How can each of these rates be interpreted as the slope of the tangent to a curve?

CHAPTER IV

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

18. Introduction.—The method of the last chapter may of course be applied to any function for the purpose of finding its derivative. It is the fundamental method since it is a direct application of the *definition* of the derivative. The procedure is however rather tedious and it is natural that we should try to develop some general rules which will enable us to arrive at the result more easily. Thus, after finding that the derivatives of x^2, x^3 and x^4 are, respectively, $2x$, $3x^2$ and $4x^3$, we might suspect that the derivative of x^n is nx^{n-1}—at least if n is a positive integer. It can be shown, *by applying the fundamental method of differentiation* to the function

$$y = x^n,$$

that this is true. We may then remember it as a formula and dispense with the labor of going through the formal process for each case. We might suspect also that the derivative of a function of the form kx^n is merely k times the derivative of x^n. Thus,

$$\frac{d}{dx}(5x^3) = 5 \cdot \frac{d}{dx}x^3 = 5 \cdot (3x^2) = 15x^2.$$

Another rule which is easily proved is that, if a function consists of the sum of several terms, its derivative is merely the sum of the derivatives of the separate terms. Thus, if

$$y = x^3 + 8x^2,$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(8x^2)$$

$$= 3x^2 + 16x.$$

This last rule is stated formally by saying that if u and v are any two functions of x, then

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

In the above example $u = x^3$ and $v = 8x^2$.

19. Formulas for differentiation.—We proceed now to write down and prove a list of eleven formulas which will enable us to dispense with the fundamental differentiation process in calculating the derivatives of simple algebraic functions such as those considered in the last chapter. Later on we shall add to this list the additional formulas necessary for finding the derivatives of trigonometric, exponential, and other transcendental functions.

It must be kept in mind that in these formulas *u and v stand for functions of x* while *c and n are constants.*

FORMULAS

(I) $$\frac{dc}{dx} = 0$$

The derivative of a constant is zero.

(II) $$\frac{dx}{dx} = 1$$

The derivative of a variable with respect to itself is one.

(III) $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

*The derivative of the sum of two functions is equal to the sum of their separate derivatives. This is easily extended to the case of any finite number of functions.**

(IV) $$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

* The formula is not necessarily true in the case of an *infinite series.* The situation is discussed briefly in Chap. XXVII.

The derivative of the product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.

(IVs)* $$\frac{d}{dx}(c \cdot v) = c\frac{dv}{dx}$$

The derivative of the product of a constant and a function is the constant times the derivative of the function.

(V) $$\frac{d}{dx}v^n = nv^{n-1}\frac{dv}{dx}$$

The derivative of the nth *power of a function is n times the* $(n - 1)$th *power of the function times the derivative of the function.*

(Vs) $$\frac{d}{dx}x^n = nx^{n-1}$$

The derivative of the nth *power of x is n times the* $(n - 1)$th *power of x.*

(VI) $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The derivative of the quotient of two functions is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(VIs) $$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{1}{c}\frac{du}{dx}$$

The derivative of the quotient of a function by a constant is the derivative of the function divided by the constant.

(VII) $$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

If y is a function of v, and v is in turn a function of x, then the derivative of y with respect to x equals the product of the derivative of y with respect to v and the derivative of v with respect to x.

* This formula is a special case of (IV). Why?

(VIII)
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

The derivative of y with respect to x is the reciprocal of the derivative of x with respect to y.

19a. Formulas (I) to (VI).—To prove any of these formulas we may take the corresponding function and apply the fundamental process of differentiation.

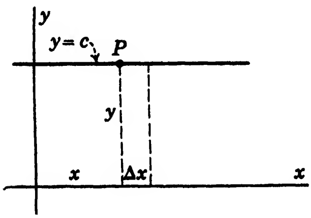

FIG. 18.

(I) Consider the function

$$y = c. \qquad \text{(Fig. 18)}$$

Starting at any point P and letting x increase by a small amount Δx we see immediately that Δy is identically 0. Hence

$$\frac{\Delta y}{\Delta x} \equiv 0, \qquad (\Delta x \neq 0)$$

and consequently

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.$$

(II) It is left for the student to sketch the function

$$y = x$$

and show that in this case

$$\Delta y \equiv \Delta x$$
$$\frac{\Delta y}{\Delta x} \equiv 1$$

and hence

$$\frac{dy}{dx} = 1.$$

(III) Consider the function

$$y = u + v$$

where u and v are differentiable functions of x. If, starting with any fixed value, x increases by a small amount Δx, u and v will change by corresponding small amounts, Δu and Δv respectively. The change produced in the value of y is

$$\begin{aligned}\Delta y &= [(u + \Delta u) + (v + \Delta v)] - (u + v)\\ &= \Delta u + \Delta v.\end{aligned}$$

Dividing Δy by Δx and then letting $\Delta x \to 0$, we have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x};$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}.$$

Hence

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \quad \text{Why?}$$

(IV) Consider the function

$$y = u \cdot v$$

and apply to it the procedure just used on the function $u + v$. We obtain

$$\begin{aligned}\Delta y &= [(u + \Delta u)(v + \Delta v)] - (uv)\\ &= u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v.\\ \frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v.\\ \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} + 0. \quad \text{Explain.}\end{aligned}$$

(IVs) This is a special case of (IV). Why? Show that (IV) reduces to this if one of the factors in the product is a constant.

(V) Consider the function

$$y = v^n$$

where v is a differentiable function of x. An increment Δx in x produces a change Δv in the value of v, and the corresponding change in y is

$$\Delta y = (v + \Delta v)^n - v^n.$$

Assuming that n is a positive integer we have, using the binomial theorem,

$$\Delta y = \left[v^n + nv^{n-1}\Delta v + \frac{n(n-1)}{2!}v^{n-2}\overline{\Delta v}^2 + \cdots + \overline{\Delta v}^n\right] - v^n$$

$$= nv^{n-1}\Delta v + \frac{n(n-1)}{2!}v^{n-2}\overline{\Delta v}^2 + \cdots + \overline{\Delta v}^n.$$

$$\frac{\Delta y}{\Delta x} = nv^{n-1}\frac{\Delta v}{\Delta x} + \frac{n(n-1)}{2!}v^{n-2}\frac{\Delta v}{\Delta x}\cdot\Delta v + \cdots + \frac{\Delta v}{\Delta x}\cdot\overline{\Delta v}^{n-1}.$$

All of the terms after the first have the limit *zero* as $\Delta x \to 0$. Why? We have then

$$\frac{dy}{dx} = nv^{n-1}\frac{dv}{dx}.$$

This proof applies only if n is a positive integer. However, *the formula is valid for all values of n.* We shall therefore use it in this sense although the proof will be left until later.*

(Vs) This is a special case of (V). Explain.

(VI) Applying to the function

$$y = \frac{u}{v}$$

the procedure used above on the function $(u + v)$ we obtain

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{v\,\Delta u - u\,\Delta v}{v(v + \Delta v)}.$$

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \quad \text{Explain.}$$

* See p. 98.

(VIs) This is a special case of (VI). Is this also equivalent to (IVs)?

Usually, in finding the derivative of a given function, one must use not one but *several* of the formulas. The details will be illustrated by four examples.

Example 1

$$y = 8x^4 + 5x^2 - 9x.$$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(8x^4) + \frac{d}{dx}(5x^2) - \frac{d}{dx}(9x) && \text{[by (III)]}\\
&= 8\frac{d}{dx}x^4 + 5\frac{d}{dx}x^2 - 9\frac{d}{dx}x && \text{[by (IVs)]}\\
&= 8(4x^3) + 5(2x) - 9(1) && \text{[by (Vs)]}\\
&= 32x^3 + 10x - 9.
\end{aligned}$$

Example 2

$$y = (4x^2 + 5)^3.$$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= 3(4x^2 + 5)^2\frac{d}{dx}(4x^2 + 5) && \text{[by (V)]}\\
&= 3(4x^2 + 5)^2(8x + 0)\\
&= 24x(4x^2 + 5)^2.
\end{aligned}$$

Example 3

$$f(x) = (2x + 1)\sqrt{x^2 + 4}.$$

Solution

$$\begin{aligned}
f'(x) &= (2x + 1)\frac{d}{dx}\sqrt{x^2 + 4} + \sqrt{x^2 + 4}\frac{d}{dx}(2x + 1) && \text{[by (IV)]}\\
&= (2x + 1)\tfrac{1}{2}(x^2 + 4)^{-\frac{1}{2}}\frac{d}{dx}(x^2 + 4) + \sqrt{x^2 + 4}\cdot 2\\
&= \frac{(2x + 1)x}{\sqrt{x^2 + 4}} + 2\sqrt{x^2 + 4}\\
&= \frac{4x^2 + x + 8}{\sqrt{x^2 + 4}}.
\end{aligned}$$

Example 4

$$s = \frac{a - t}{a + t}.$$

Solution

$$\frac{ds}{dt} = \frac{(a+t)\frac{d}{dt}(a-t) - (a-t)\frac{d}{dt}(a+t)}{(a+t)^2} \qquad \text{[by (VI)]}$$

$$= \frac{(a+t)(0-1) - (a-t)(0+1)}{(a+t)^2}$$

$$= \frac{-2a}{(a+t)^2}.$$

PROBLEMS

1. How may the function $f(x) = x^2 + 3x - 4$ be regarded as the sum of two functions u and v? Can its graph be obtained by sketching $y = x^2$ and $y = 3x - 4$ and adding ordinates? How may it be regarded as the sum of three functions?

2. Find the derivative of $y = x(x^2 + 4)$ both with and without the use of formula (IV). Explain why formula (IV) would be used in finding the derivative of the function $x \sin x$. What else would we have to know?

3. Which formulas out of the given list would be required in finding the derivative of $\varphi(x) = \frac{\log x}{x^2}$? What additional information would be necessary? How do you think this information might be obtained?

4. Differentiate the function $y = (x^2 + 1)^3$ with and without the use of formula (V). Why is the derivative *not* equal to $3(x^2 + 1)^2$?

5. Show how formula (III) may be extended to the sum of any number of functions. HINT: $u + v + w$ may be written $(u + v) + w$.

Differentiate the following functions:

6. $y = 2x^2 - 7x + 1.$

7. $y = x^3 - 6x.$

8. $s = 4t^3 - 8t + 3.$

9. $w = (2t + 1)^2.$

10. $y = (a^2 - x^2)^{\frac{1}{2}}.$

11. $y = \sqrt{4x}.$

12. $y = \sqrt{\frac{1}{x}}.$

13. $y = 4x^{\frac{3}{2}}.$

14. $z = (2 - 3y)^{-\frac{1}{2}}.$

15. $y = x^3 + \frac{32}{x}.$

16. $y = \frac{3x + 4}{x}.$

17. $y = \frac{x^2 + 5}{x - 1}.$

18. $s = \frac{1}{t^2 + 4}.$

19. $y = x\sqrt{x^2 + 4}.$

20. $y = \sqrt{2 + \frac{1}{x}}.$

21. $y = (x + 2)\sqrt{4x - 1}.$

22. $y = \sqrt{\frac{1 + x^2}{1 - x^2}}.$

23. $w = \frac{y}{\sqrt{1 - 4y^2}}.$

24. $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

25. $y = (a^{\frac{1}{2}} - x^{\frac{1}{2}})^2$.

26. $y = \frac{8a^3}{x^2 + 4a^2}$.

27. $y = \left(\frac{x^2 - 4}{x^2 + 4}\right)^2$.

28. $w = \frac{4t}{t^2 + 4}$.

29. $y = \frac{4}{x} + \frac{1}{1 - x}$.

In each of the following, find dy/dx after first solving the given relation for y in terms of x. Explain why the double sign ($\pm$) must be used in some cases.

30. $4x = 2y + 7$.

31. $ax + by + c = 0$.

32. $x + 4 = 2y^2$.

33. $x^2 + y^2 = r^2$.

34. $y^2 = 4px$.

35. $xy = 16$.

36. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

37. $2(y - 5) = (x + 2)^2$.

Find the slope of the tangent line to each of the following curves at the point indicated. Sketch the curve and tangent line.

38. $y = \frac{8}{x^2}$, at $x = 2$.

39. $xy = 2$, at $x = 1$.

40. $3x + 4y = 12$, at $x = 4$.

41. $y^2 = 16x$, at $(4, 8)$ and $(4, -8)$.

42. $x^2 + y^2 = 16$, at $(3, -\sqrt{7})$.

43. $\frac{x^2}{16} + \frac{y^2}{9} = 1$, at $\left(2, \frac{3\sqrt{3}}{2}\right)$.

44. Show that if u, v, and w are differentiable functions of x,

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}.$$

HINT: Apply the product formula to $(uv)(w)$.

Differentiate each of the following functions using the formula of Prob. 44. Check by multiplying out the right-hand side and then differentiating.

45. $y = x(x - 1)(x - 2)$.

46. $y = x(x^2 + 3)(2x - 3)$.

47. $y = (4x + 1)(x^2 - 3)(x - 2)$.

48. Suppose we sketch the curves $y = f(x)$ and $y = \varphi(x)$, and then obtain the product curve $y = f(x) \cdot \varphi(x)$ by multiplying ordinates. Is the slope of the product curve at $x = a$ equal to the product of the slopes of the component curves or what is the relation?

49. Show that if the tangent to each of the component curves of Prob. 48 is horizontal at $x = a$, then the same is true of the tangent to the product curve.

50. Find the rates of change of both the volume v and surface area s of a sphere with respect to the radius. If the radius is in inches, in what units are these rates expressed?

51. Find the rate of change of the volume of a sphere with respect to its surface area using the results of Prob. 50 and also by expressing v as a function of s and differentiating.

52. Find the rate of change of the area of an equilateral triangle with respect to a side. If the side is expressed in feet, what are the units of this rate?

53. Assuming that the edge x of a cube is increasing, find the rates of change of the volume v and the surface area s with respect to the edge. Find the numerical values of these rates when $x = 6$ in. and state the units in which they are expressed. If the edge is increasing at 2 in. per minute, find the *time* rates at which v and s are increasing.

20. Differentiation of a function of a function. Formula (VII).—Suppose that y is expressed, not directly in terms of x, but as a function of another variable v which is in turn a function of x. Thus, we may have

$$y = v^2 - 3v + 2, \qquad v = 4x^2 + 1.$$

If we regard x as the independent variable it is clear that y is a function of x through v, and we could express y directly in terms of x by eliminating v. A method of obtaining dy/dx without doing this is contained in the formula

$$\text{(VII)} \qquad \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

Thus, in the above example,

$$\frac{dy}{dv} = 2v - 3 \qquad \text{and} \qquad \frac{dv}{dx} = 8x$$

Using (VII) we have

$$\begin{aligned} \frac{dy}{dx} &= (2v - 3) \cdot (8x) \\ &= 8x(8x^2 - 1). \end{aligned}$$

In order to prove the formula we need merely to note that for any value of x for which v is a differentiable function of x

and y is in turn a differentiable function of v, the quantities

$$\frac{\Delta y}{\Delta x} \quad \text{and} \quad \frac{\Delta y}{\Delta v} \cdot \frac{\Delta v}{\Delta x}$$

are two functions of Δx which are *identically equal* for all values of $\Delta x (\neq 0)$; their limits, as Δx approaches zero, must therefore be equal.* These limits are respectively dy/dx and $dy/dv \cdot dv/dx$; hence,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

An interesting special case is that in which x and y are connected by a relation $y = f(x)$, and x is known to vary with the time according to some law $x = \varphi(t)$. The rate at which x is changing at any instant (with respect to time) is given by dx/dt; if we wish to find the time rate of change of y, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

The student should notice that, written in the form

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}, \qquad \left(\frac{dx}{dt} \neq 0\right)$$

the formula merely states that the rate of change of y, *measured with respect to* x, is equal to the quotient of the rates at which y and x are changing with respect to the time.

21. Differentiation of inverse functions. Formula (VIII). We may have y defined as a function of the independent variable x by an equation of the form

$$x = \varphi(y).$$

If we solve this equation for y in terms of x, we obtain the "inverse" function $y = f(x)$. It may however be difficult

* It should be specified also that Δv should not be zero. The student should consider the case in which $\Delta v = 0$.

or even impossible to solve for y, and a method of calculating dy/dx without this preliminary step is contained in the formula

$$\text{(VIII)} \qquad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$

Example

Compute dy/dx for the curve $x = y^3 - 4y$.

Solution

Considering y as the independent variable, we have

$$\frac{dx}{dy} = 3y^2 - 4.$$

If we wish now to consider x the independent variable we have, using (VIII),

$$\frac{dy}{dx} = \frac{1}{3y^2 - 4}.$$

The student should sketch the curve for the above example, noting carefully that while the given relation defines x as a single-valued function of y when y is taken as the independent variable, it defines y as a *triple-valued* function of x (for some values of x) when x is taken as the independent variable. Proper care must therefore be taken to restrict the discussion to one branch of the function at a time.

The proof of the formula follows immediately from the fact that the quantities

$$\frac{\Delta y}{\Delta x} \quad \text{and} \quad \frac{1}{\dfrac{\Delta x}{\Delta y}}$$

are identically equal for all values of Δx and $\Delta y (\neq 0)$ and hence must have the same limit as Δx and Δy approach zero simultaneously, no matter which variable is considered independent, if the limits exist and are $\neq 0$.

PROBLEMS

In each of the followinτ, find dy/dx in terms of x both with and without eliminating v:

1. $y = v^2 + 2,\ v = 2x - 5.$

2. $y = v^2 - 3v + 1,\ v = x^2 + 1.$

3. $y = \dfrac{v^2}{v^2 - 1},\ v = \dfrac{1}{x}.$ **4.** $y = \sqrt{\dfrac{4}{v}},\ v = x^2.$

In each of the following, y is assumed to vary with x according to the law expressed by the first equation, while x varies with the time t as indicated by the second; in each case find dy/dt in terms of t:

5. $y = \dfrac{x+1}{x-1},\ x = 2t + 1.$ **6.** $y = \dfrac{8}{x^2 + 4},\ x = 2\sqrt{t}.$

7. $y = \dfrac{4x}{x^2 + 1},\ x = t^2.$ **8.** $y = x^3 - 9x,\ x = \frac{1}{2}(t + 2).$

9. A point moves along the line $y = 3x - 5$ so that $x = 2t$. At what rate is y changing when the moving point goes through $P(6, 13)$?

10. A point moves along the parabola $y = x^2 - 2x - 15$ so that $x = 2t$ where t is the time. Compute the time rate of change of y when $t = 0, \frac{1}{2}$, and 2. Sketch the curve and describe the motion.

11. A point moves along the hyperbola $y = 6/x$ so that $x = 2/t$. Is y increasing or decreasing and at what time rate when $t = 2$? Sketch the curve and describe the motion.

12. If y is changing three times as rapidly as x, then of course x is changing $\frac{1}{3}$ as rapidly as y. What connection does this statement have with the formula $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$?

13. Sketch a curve to represent $y = f(x)$. Show the geometrical significance of the value of dx/dy at a point P on this curve. HINT: Starting at P give an increment Δy to y; show the change Δx produced in x; study the meaning of $\Delta x/\Delta y$.

14. Sketch the curve $x = y^3 - 9y$ and compute the value of dy/dx at the points where it crosses the y-axis.

15. Sketch the curve $x = \dfrac{2y + 8}{y + 1}$. Compute the value of dy/dx at the points where it crosses the axes, both with and without solving for y.

16. Compute the rate of change of the volume of a sphere with respect to its surface area. If the radius of a sphere is increasing, when is the volume (in cubic inches) increasing at the same rate as the surface area (in square inches)?

17. Write out a formal proof of the formula

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx}.$$

Discuss the case in which $\Delta v \equiv 0$.

18. Write out a formal proof of the formula

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$

Discuss its limitations.

22. Implicit functions.—The relation between two variables x and y is often expressed most conveniently by means of an equation of the form

$$\varphi(x, y) = 0.$$

If x is regarded as the independent variable such an equation is said to define y as an *implicit* function of x. It "implies" that an explicit relation of the form

$$y = f(x)$$

exists, and might be obtained by solving for y in terms of x. Since, however, it may be difficult or even impossible to solve for y, we need a method of computing dy/dx directly from the implicit form of the relation.

The method consists of differentiating each term in the equation *with respect to* x, and then solving the resulting equation for dy/dx.

Example

Compute dy/dx from the equation $x^3 + y^3 = xy + 4$.

Solution

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(xy) + \frac{d}{dx}(4)$$

$$3x^2 + 3y^2\frac{dy}{dx} = \left(x\frac{dy}{dx} + y\right) + 0.$$

Solving this equation for dy/dx we have

$$\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}.$$

In using this procedure the student must be very careful to differentiate each term *with respect to x*, noting for example that

$$\frac{d}{dx}(y^3) = \text{not } 3y^2, \text{ but } 3y^2\frac{dy}{dx}.*$$

In order to see more clearly the justification for the method used here, let us suppose that we might solve the equation

$$x^3 + y^3 = xy + 4$$

for y in terms of x and then substitute this result back into the equation for y. The resulting equation would obviously be an *identity* in x; *i.e.*, the two sides of the equation would be identically the same. The derivative of the left side would then of course be identically equal to the derivative of the right side. In our procedure we are equating the derivatives of the two sides with respect to x without solving for y in terms of x. We are regarding y as the unknown function of x which would reduce the equation to an identity.

PROBLEMS

In each of the following, find dy/dx both with and without solving for y in terms of x. Show that the results are equivalent:

1. $3x + 8y = 24$.
2. $ax + by = c$.
3. $x^2 - y^2 = 10$.
4. $y^2 = 16x$.
5. $x^2 + y^2 = r^2$.
6. $y^3 = x^2$.
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

In each of the following, find dy/dx:

9. $x^2 + xy - 4y^2 = 1$.
10. $x^2 + y^2 - 8x - 24y = 10$.
11. $y^2 + 4 = 2x$.
12. $x^2 + 4y^2 + 4y = 32$.
13. $xy = (x + y)^2$.
14. $y^2 - 2x^2 = 4 + 3xy$.
15. $3x^2 + 8xy - 3y^2 - 4x - y = 6$.
16. $x^3 + y^3 = 3axy$.
17. $x^3 + 4y^3 = 6 - 4xy^2$.
18. $x^3 = 4y^3 + 6xy$.
19. $4x^2y + 2y^3 = 1 + xy^2$.
20. $x^4 - 2y^4 = x^2y^2 - 6$.

* Note that $3y^2$ is the derivative of y^3 *with respect to y*. In order to get the derivative *with respect to x* we must multiply this result by dy/dx in accordance with the formula $du/dx = du/dy \cdot dy/dx$.

CHAPTER V

APPLICATIONS OF THE DERIVATIVE

23. Angle of intersection of curves.—If the two curves $y = f(x)$ and $y = F(x)$ intersect at a point P, their angle of intersection is defined to be the angle φ between the tangents drawn to the curves at P (Fig. 19).

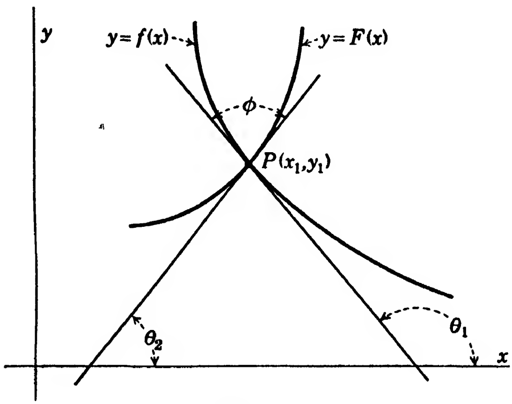

FIG. 19.

This angle is found easily as follows:

1. Determine the coordinates (x_1, y_1) of a point P of intersection by solving the equations simultaneously.

2. Find the slope of the tangent to each curve at P. Call these slopes m_1 and m_2.

3. From the figure it is evident that

$$\varphi = \theta_1 - \theta_2;$$

hence,

$$\tan \varphi = \tan (\theta_1 - \theta_2)$$

or

$$\tan \varphi = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

If there are several points of intersection, each one must of course be considered separately.

Example

At what angle do the parabolas $y = x^2/8$ and $y^2 = x$ intersect?

Solution (Fig. 20)

1. Solving the equations we find that the curves intersect at $P(4, 2)$. [We disregard the obvious 90° intersection at (0, 0).]

2. We next find the slope of each curve at P as follows:

$$y = \frac{x^2}{8} \qquad\qquad y^2 = x$$

$$\frac{dy}{dx} = \frac{x}{4} \qquad\qquad 2y\frac{dy}{dx} = 1$$

$$m_1 = \frac{dy}{dx}\bigg]_{4,2} = 1. \qquad\qquad \frac{dy}{dx} = \frac{1}{2y}$$

$$m_2 = \frac{dy}{dx}\bigg]_{4,2} = \frac{1}{4}.$$

3. The tangent of the required angle is then,

$$\tan \varphi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1 - \frac{1}{4}}{1 + 1 \cdot \frac{1}{4}} = \frac{3}{5}.*$$

$$\varphi = \text{arc } \tan \frac{3}{5} = 30° \, 58'.$$

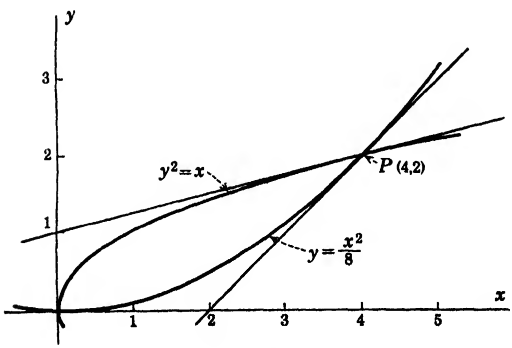

FIG. 20.

24. Equations of tangent and normal.—The student will recall from his study of analytic geometry that the equation

* If we interchange m_1 and m_2, taking $m_1 = \frac{1}{4}$ and $m_2 = 1$, we get $\tan \varphi = -\frac{3}{5}$. The positive result corresponds to the acute angle and the negative one to its supplement.

of the line which passes through a given point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1).$$

Using this, one can easily write down the *equation* of the line which is tangent to the curve $y = f(x)$ at a given point $P(x_1, y_1)$ on the curve. The value of dy/dx at P is the slope of the required line, and of course the given point P is one point on it.

The line through P perpendicular to the tangent line is called the *normal line;* since its slope is $-1/m$, its equation can also be easily written down using the above "point-slope" form.

Example

Write the equation of the line which is tangent to the parabola $y^2 = x$ at $P(4, 2)$.

Solution (Fig. 20)

$$\frac{dy}{dx} = \frac{1}{2y}; \qquad \left.\frac{dy}{dx}\right]_{4,2} = \tfrac{1}{4}.$$

The equation of the line through $P(4,2)$ with slope $\frac{1}{4}$ is

$$y - 2 = \tfrac{1}{4}(x - 4)$$

or

$$4y - x = 4.$$

PROBLEMS

1. Sketch the circle $x^2 + y^2 = 25$ and the line $2y - x = 5$. Determine the points of intersection. Find the angle of intersection at each of these points.

2. Sketch the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$. Find the points of intersection and the angles of intersection.

3. At what angles do the parabolas $y^2 = x$ and $x^2 = y$ intersect?

4. At what angle does the hyperbola $xy = 4$ cut the circle $x^2 + y^2 = 17$?

5. Show that the hyperbolas $x^2 - y^2 = a^2$ and $xy = a$ intersect at right angles.

6. Sketch the curves $y = \dfrac{8}{x^2 + 4}$ and $2y^2 = x$. Find their angle of intersection.

7. Find the angles at which the line $y = x - 1$ cuts the parabola $y = x^2 - 6x + 5$.

Sketch each of the following curves. Draw tangent and normal lines at the point indicated. Write the equations of these lines.

8. $xy = 12$ at $(3, 4)$.

9. $x = y^2 - y - 2$ at $(0, 2)$.

10. $y = x^3 + 2$ at $(1, 3)$.

11. $x^2 + 4y^2 = 25$ at $(3, 2)$.

12. $y = x^2 - 2x - 8$ at $(4, 0)$ and $(1, -9)$.

13. What is the equation of the line with slope 6 which is tangent to the curve $y = x^2 - 4x - 5$?

14. A line is parallel to the line $3x + y = 2$ and is also tangent to the curve $y = x^2 + x - 6$. What is its equation?

15. Find the equation of the line with slope $\frac{1}{2}$ which is normal to the parabola $y = x^2 - 4x - 5$.

16. Show that the equation of the line tangent to the circle $x^2 + y^2 = r^2$ at (x_1, y_1) is $x_1x + y_1y = r^2$.

17. Show that the equation of the line tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (x_1, y_1) \text{ is } \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

18. Sketch the curve $y = x^3 - 6x^2 + 3x + 10$. A line is drawn tangent to this curve at $(3, -8)$. Find the angle at which this line intersects the curve at the other point of intersection.

25. Time rates.—We have emphasized the fact that the derivative of a function $f(x)$ with respect to x is the rate of change of $f(x)$, *measured with respect to* x. Suppose now that we have a variable quantity Q which is known to vary with the time according to some definite law

$$Q = f(t).$$

The value of dQ/dt at any instant is evidently the rate at which Q is changing at that instant, *measured with respect to the time.* If Q were expressed for example in pounds, and t in minutes, the value of dQ/dt would be the rate of change of Q in pounds per minute.

A purely artificial example which will make the idea clear is as follows: Suppose that water is being run into a tank in such a way that the number of gallons in the tank at the end of t minutes is given by the equation

$$G = 4 + \sqrt{t + 1}.$$

The rate at which water must be entering the tank at any time t is

$$\frac{dG}{dt} = \frac{1}{2\sqrt{t+1}} \text{ gal. per minute.}$$

At the end of say 15 min., the tank would contain 8 gal., and water would be entering at a rate of

$$\left.\frac{dG}{dt}\right]_{t=15} = \tfrac{1}{8} \text{ gal. per minute.}$$

26. Velocity in rectilinear motion.—Let a particle P move along the straight line AB in such a way that its distance

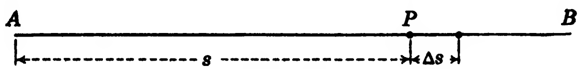

FIG. 21.

from A at the end of time t is (Fig. 21)

$$s = f(t).$$

In a small additional time Δt it moves a corresponding small distance Δs, and its average velocity over this distance is expressed by the fraction

$$\frac{\Delta s}{\Delta t}. \quad \text{Why?}$$

The instantaneous velocity of the particle is defined as the limit of this quotient as Δs and Δt approach zero simultaneously; that is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

As an interesting special case we may consider the motion of a projectile which is thrown vertically upward from a point A with initial velocity v_0; its distance from A at the end of t seconds is known to be (neglecting air resistance),

$$s = v_0 t - \tfrac{1}{2}gt^2. \qquad (g = 32.2.)$$

Its velocity at any time t, in feet per second, is given by

$$\frac{ds}{dt} = v_0 - gt.$$

27. Related rates.—Suppose that two variables x and y are connected by a relation $y = f(x)$, and that x changes with the time at a known rate. The corresponding time rate of change of y can be found from the relation

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Example 1

A ladder 25 ft. long rests against a wall and the lower end is being pulled out from the wall at 6 ft. per minute. At what rate is the top descending at the instant when the lower end is 7 ft. from the wall?

Solution (Fig. 22)

The variables x and y are connected by the relation

$$x^2 + y^2 = 25^2.$$

We are given $dx/dt = 6$, and must find the value of dy/dt at the instant when $x = 7$. Differentiating the relation between x and y we find first that

$$\frac{dy}{dx} = -\frac{x}{y};$$

Hence,

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

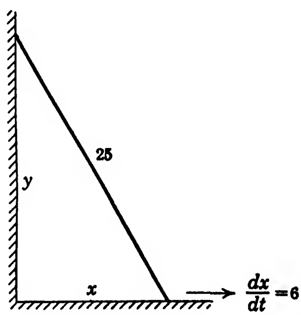

FIG. 22.

At this instant $x = 7$, $y = 24$, and $dx/dt = 6$; hence

$$\frac{dy}{dt} = -\frac{7}{24} \cdot 6 = -\frac{7}{4} \text{ ft. per minute.}$$

The negative sign indicates that y is *decreasing*.

Example 2

A cone is 10 in. in diameter and 10 in. deep. Water is poured into it at 4 cu. in. per minute. At what rate is the water level rising at the instant when the depth is 6 in.?

Solution (Fig. 23)

We must first express the volume V of water in the cone in terms of its depth y. We are given that $dV/dt = 4$ cu. in. per minute and must compute the value of dy/dt at the instant when $y = 6$ in.

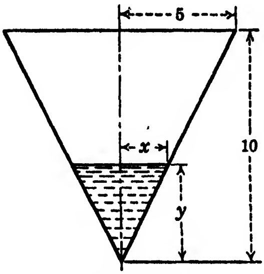

FIG. 23.

$$V = \tfrac{1}{3}\pi x^2 y;$$

but

$$\frac{x}{y} = \frac{5}{10}, \qquad \text{or} \qquad x = \tfrac{1}{2}y;$$

therefore,

$$V = \frac{\pi y^3}{12}.$$

$$\frac{dV}{dy} = \frac{\pi y^2}{4};$$

$$\frac{dV}{dt} = \frac{\pi y^2}{4}\frac{dy}{dt}.$$

At this instant $y = 6$ and $dV/dt = 4$; hence,

$$\frac{dy}{dt} = \frac{4}{9\pi} \text{ in. per minute.}$$

PROBLEMS

1. A small lead ball is thrown vertically upward from a point A on the ground with initial velocity 120 ft. per second. Find:

(*a*) Its velocity at the end of 2 sec.; 4 sec.

(*b*) Its greatest distance from the ground.

(*c*) The total time in the air.

2. A small heavy object is dropped from the top of a cliff 600 ft. high. With what velocity will it strike the ground below?

3. Solve Prob. 2 for the case in which the object is thrown *downward* with initial velocity 60 ft. per second; for the case in which it is thrown *upward* with this velocity.

4. A particle moves along the x-axis so that its distance from the origin at the end of t min. is $x = 2 + \sqrt{t + 1}$. Describe the motion. Find the position and velocity of the particle at the end of 8 min.

5. A man walking across a bridge at 6 ft. per second observes at a certain instant that a boat is passing directly under him. The bridge is 30 ft. above the water; the boat is traveling at 12 ft. per second. At what rate is the distance s between the man and boat changing 2 sec. later?

6. A boat B is 12 miles west of another boat A. B starts east at 8 m.p.h. and at the same time A starts north at 12 m.p.h. At what rate is the distance between them changing at the end of $\frac{1}{2}$ hr.?

7. The legs a and b of a right triangle are 4 in. and 6 in. respectively. At the same instant, a starts increasing at 2 in. per minute and b starts decreasing at 1 in. per minute. Express the area of the triangle after t min. as a function of t. Is the area increasing or decreasing at the end of 1 min. and at what rate? What is the situation at the end of 4 min.? Is there any instant at which the area is not changing? Would the area at this instant be a maximum?

8. Sketch the function $y = \sqrt{625 - x^2}$. Explain how Example 1 (page 51) may be considered as a problem on the motion of a point along this curve.

9. The side of a square is increasing at 2 in. per minute. At what rate is the area increasing when each side is 8 in. long?

10. If the radius of a sphere is 10 in. and is decreasing at the rate of 1 in. per minute, at what rate is the volume decreasing?

11. A street light hangs 18 ft. above the street. A man 6 ft. tall walks away from it at 5 m.p.h. At what rate does the farther end of his shadow move? At what rate does his shadow lengthen?

12. A point moves along the parabola $y^2 = 16x$. At what rate is its distance from the focus changing when it passes through (1, 4) if x is increasing at 8 units per minute? Could you have anticipated this result from the definition of a parabola?

13. A point moves along the parabola $y = x^2$, its abscissa increasing uniformly at 2 units per minute. At what rate is its distance from $(-1, 1)$ changing when it passes through (2, 4)?

14. A funnel is 12 in. in diameter and 8 in. deep. Water is poured in at 20 cu. in. per minute. At what rate will the surface be rising at the instant when it begins to overflow if water is flowing out at the bottom of the funnel at 4 cu. in. per minute?

15. A water trough is 8 ft. long and 4 ft. deep. Its cross section is a trapezoid 2 ft. wide at the bottom and 4 ft. wide at the top. At what rate must water be poured in to cause the surface to rise at 3 in. per minute when the depth is 1 ft.?

16. A man standing on a dock 40 ft. above the water pulls a boat toward the dock by taking in rope at 4 ft. per second. At what rate is the boat approaching the dock when 80 ft. of rope are out?

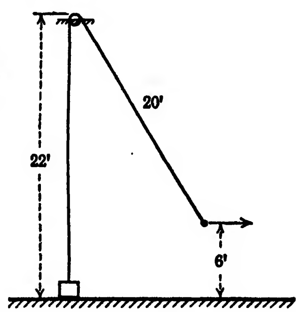

FIG. 24.

17. A weight is attached to a rope 42 ft. long. The rope passes over a pulley 22 ft. above the ground. A man takes hold of the end of the rope and walks back at 10 ft. per second holding the end at a level of 6 ft. above the ground. At what rate is the weight ascending when it is 6 ft. above the ground? See Fig. 24.

18. A cistern has the form of a frustum of a cone. It is 12 ft. deep and the diameter is 8 ft. at the top and 4 ft. at the bottom. Water is being run in at 10 cu. ft. per minute. When the water is 6 ft. deep the surface is observed to be rising at 3 in. per minute. At what rate is water seeping into the banks?

28. Maxima and minima.—The definitions of maximum and minimum values of a function were given in Chap. I. We shall see now how to determine such values exactly.

Suppose that the equation of the curve shown in Fig. 25 is $y = f(x)$. The value of dy/dx at any point is the slope of the tangent to the curve at that point. If we should set *dy/dx equal to zero* and solve for x, we would evidently

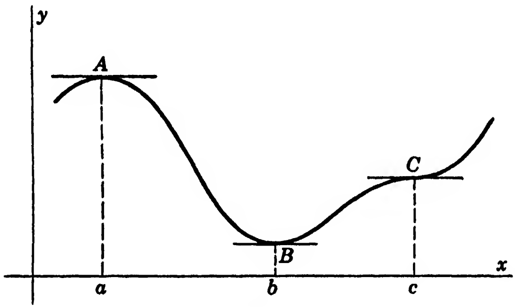

FIG. 25.

obtain the abscissas of the points at which the tangent line is *horizontal; i.e.,*

$$\frac{dy}{dx} = 0 \text{ at } x = a,\ x = b, \text{ and } x = c.$$

From the points so obtained we can easily separate the ones which give maximum or minimum values of y as follows:

If the point where $x = a$ is a *maximum* point, then dy/dx is *positive* for x slightly less than a, and *negative* for x slightly more than a.

If the point where $x = b$ is a *minimum* point, then dy/dx is *negative* for x slightly less than b, and *positive* for x slightly more than b.

If the point where $x = c$ is *neither* a maximum nor a minimum point, then dy/dx has the *same* sign for x slightly more than c as it has for x slightly less than c.

The details will be made clear by two examples which the student should study carefully.

Example 1

Find the maximum and minimum points on the curve

$$y = x^3 - x^2 - 8x + 6.$$

Solution

$$\frac{dy}{dx} = 3x^2 - 2x - 8.$$

Setting $dy/dx = 0$ and solving for x we have

$$3x^2 - 2x - 8 = 0$$
$$(3x + 4)(x - 2) = 0$$
$$x = 2, \text{ or } -\tfrac{4}{3}.$$

The tangent line is then horizontal at the points where $x = 2$ and

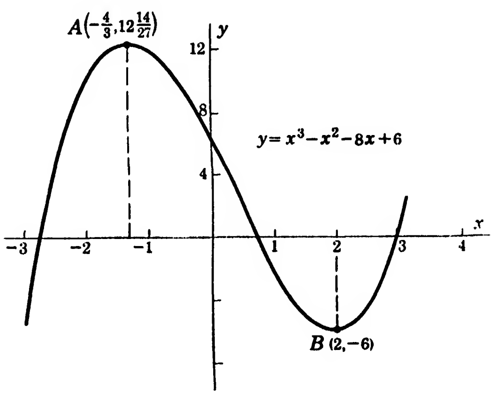

FIG. 26.

$x = -\frac{4}{3}$. To determine whether $x = 2$ gives a maximum or minimum point on the curve we write dy/dx in the factored form:

$$\frac{dy}{dx} = (3x + 4)(x - 2).$$

The first factor is obviously $+$ for all values of x near 2. However,

If x is slightly *less* than 2, $(x - 2)$ is $-$ and dy/dx is $-$.

If x is slightly *more* than 2, $(x - 2)$ is $+$ and dy/dx is $+$. Since the value of dy/dx is zero *at* $(2, -6)$ and changes from $-$ to $+$ when we go through this point from left to right, the point is a *minimum* point. Similarly, it may be shown that $(-\frac{4}{3}, 12\frac{14}{27})$ is a maximum point. The curve is shown in Fig. 26.

Example 2

Examine the function $y = x^3 - 3x^2 + 3x + 1$ for maximum and minimum values.

Solution

$$\frac{dy}{dx} = 3x^2 - 6x + 3$$
$$= 3(x - 1)^2.$$

Setting $dy/dx = 0$ and solving for x, we of course get

$$x = 1.$$

The tangent line is then horizontal at $(1, 2)$. In this case the value of dy/dx is obviously $+$ when x is either more or less than 1. The point is therefore neither a maximum nor a minimum point. The curve is shown in Fig. 27. The point P is called an *inflection point*. Such points will be considered in greater detail in the next chapter.

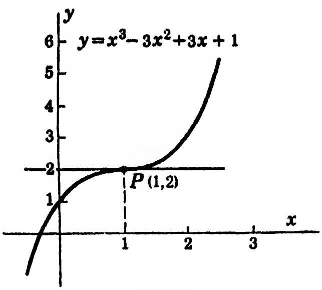

FIG. 27.

PROBLEMS

Find the coordinates of the maximum and minimum points on each of the following curves. Sketch the curve. State for what values of x the function is increasing with x and for what values it is decreasing.

1. $f(x) = x^2 - 8x + 12.$	**2.** $y = -x^2 + 4x - 7.$
3. $y = ax^2 + bx + c.$	**4.** $y = 8x^2 - x^4.$
5. $s = t^3 - 6t^2 + 9t + 1.$	**6.** $27y = x^3 - 6x^2 - 15x + 19.$
7. $v = 3u^5 - 20u^3 + 4.$	**8.** $A = 2r^2 + \frac{32}{r}.$
9. $\varphi(x) = x^3 + \frac{48}{x}.$	**10.** $D = \frac{16}{x} + \frac{4}{1 - x}.$
11. $y = \sqrt{16 - x^2}.$	**12.** $s = \frac{y^2}{y + 2}.$

13. $w = \dfrac{4t}{t^2 + 4}$.

14. $y = \dfrac{x^2 - 7x + 16}{x - 4}$.

15. $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$.

16. $\varphi(x) = (x - 2)^2(x + 3)^2$.

17. $y = (x + 6)\sqrt{x - 6}$.

18. $y = x^3 - 6x^2 + 12x - 8$.

19. $z = \dfrac{w^2 + 2w + 10}{w - 3}$.

20. $y = \dfrac{x^3}{2x + 4}$.

21. $y = x\sqrt{16 - x^2}$.

29. Other types of maxima and minima.—Before taking up the applied problems it is necessary to point out that in many important cases the determination of maximum and minimum values of a function is not a problem of calculus

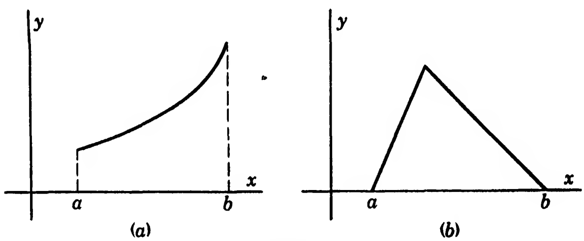

FIG. 28.

at all. One is often concerned, for example, with a function which is defined only in a limited interval from $x = a$ to $x = b$. The maximum value may occur at one end of the interval, as illustrated in Fig. 28*a*. Or the function may vary over the interval in some such way as indicated in Fig. 28*b*. In either case, the derivative is obviously *not* zero at the maximum point.

The student must understand that it is only under rather special conditions that finding a maximum or a minimum value of a function involves the problem of locating a point where the derivative is zero.

30. Applications of maxima and minima.—The problems of the next set illustrate the many cases in which the determination of extreme values of a function *is* a problem of calculus. In solving a particular problem the student should first read the problem very carefully and, if possible,

draw a figure to illustrate the situation; then proceed as follows:

1. Pick out the variable quantity Q for which a maximum or minimum value is required.

2. Pick out a *single* independent variable x on which Q depends. There may be several possible choices for this variable.

3. Express Q in terms of x—say $Q = f(x)$.

4. Find the value of x which makes Q a maximum or a minimum by setting $dQ/dx = 0$ and solving for x. Find the corresponding value of Q, if required, by substituting this value of x in $Q = f(x)$.

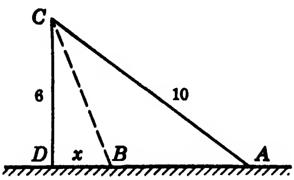

FIG. 29.

5. If necessary make the usual test to show that the result is actually a maximum value or a minimum value, whichever was required in the problem. Often the nature of the problem enables one to omit this test.

Example

A man is in a rowboat 6 miles from shore. He wishes to reach a point A on the shore 10 miles away (Fig. 29). He can row 2 m.p.h. and walk 4 m.p.h. Where should he land in order to reach A in the least time?

Solution

1. Quantity to be made a minimum: time required to go from C to A. Call this T.

2. Independent variable: We assume that he may land at any point B between D and A and let $DB = x$ be the independent variable.

3. He rows a distance $CB = \sqrt{x^2 + 36}$ at 2 m.p.h. and then walks a distance $BA = 8 - x$ at 4 m.p.h. The total time is then

$$T = \frac{\sqrt{x^2 + 36}}{2} + \frac{8 - x}{4}.$$

4. $$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 36}} - \frac{1}{4} = 0$$

$$x = 2\sqrt{3}$$
$$T = 4.60 \text{ hr. (approximately).}$$

5. Without a formal test the student can easily see that this is a minimum and not a maximum value of T.

The way in which T varies with x is shown graphically in Fig. 30. The point $B(2\sqrt{3},\ 4.60)$ is the minimum point on the curve. It is obvious that one could always represent the relation between the variables by a graph and interpret his problem as one of finding maximum or minimum points on this curve.

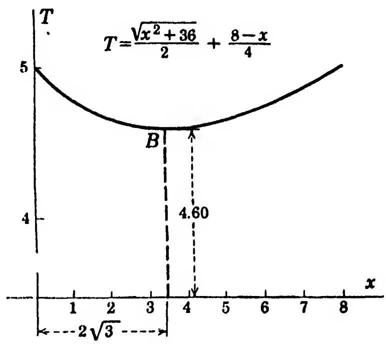

FIG. 30.

PROBLEMS

1. Find the volume of the largest box that could be made from a sheet of tin 4 ft. long and 2.5 ft. wide by cutting a square out of each corner and turning up the sides as indicated in Fig. 4, Chap. I. Illustrate by drawing a graph.

2. A rectangular box to contain 108 cu. ft. is to be made with a square base. The cost of material for bottom, top, and sides, is 1, 5, and 6 cents per square foot, respectively. Determine the dimensions for minimum cost. Illustrate by a graph. See Example 2, page 8.

3. Divide the number 12 into two parts so that the sum of the squares of the parts will be as small as possible.

4. Show that the largest rectangle that can be inscribed in a circle is a square.

5. A gardener wishes to lay out a rectangular plot with one edge along a neighbor's lot. The gardener is to pay for the fence for three sides on his own ground and for half of that along the dividing line. What dimensions would give him the least cost if the plot must contain 363 sq. rd.?

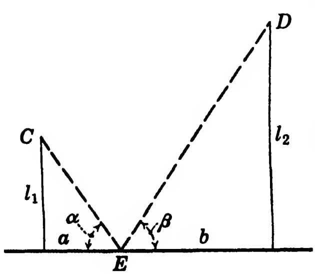

FIG. 31.

6. The straight shore of a large lake runs east and west. A and B are two points on this shore 12 miles apart. There is a town C, 9 miles north of A, and another town D, 15 miles north of B. A single pumping station on the lake shore is to supply water to both towns. Where should it be located in order that the sum of its distances from C and D may be a minimum?

7. A ray of light from C (Fig. 31) strikes a plane mirror and is reflected through D. Show that if $CE + ED$ is a minimum, angles

α and β are equal. Compare this with Prob. 6. HINT: Find a and b and show that $l_1/a = l_2/b$.

8. Find the dimensions of the cylinder of largest volume that can be inscribed in a cone of radius a and height b.

9. A ship B is 40 miles due east of a ship A. B starts due north at 25 m.p.h. and at the same time A starts in the direction north $\theta°$ east where $\theta = \text{arc tan } \frac{4}{3}$ at 15 m.p.h. When will they be closest together?

10. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.

11. What is the ratio of the volume of a sphere to that of the smallest circumscribing right circular cone?

12. A page is to contain A sq. in. of printed matter and is to have margins a inches along the sides, and b inches at the top and bottom. Show that the area of the page will be a minimum if the dimensions of the printed part are $\sqrt{aA/b}$ and $\sqrt{bA/a}$.

13. A cylindrical measure with volume V is to be made. Show that the amount of metal required will be least if its radius and height are equal.

14. Determine the length of the shortest line that has its ends on the coordinate axes and is tangent to the hyperbola $xy = 16$.

15. A real estate operator has a building containing 60 apartments. He estimates that he could keep all of them rented at \$40 per month but for each dollar added to the rent he will have one vacancy. What rental would yield the greatest gross income?

16. If the operator in Prob. 15 estimates that he saves \$4 per month in operating costs for each vacant apartment, what is the most favorable rental price?

17. A telephone company has 10,000 phones in a certain city where the rate is \$4 per month. The officials believe that if the charge is reduced, the number of telephones in use will increase at an estimated rate of 1,000 phones for each 25 cent reduction. What rate would yield the greatest gross income?

18. What would be the most favorable charge in Prob. 17 if it must be assumed that each added telephone will increase operating costs by 50 cents per month?

19. At any distance x from a source of light the intensity of illumination varies directly as the intensity of the source and inversely as the square of x. Suppose that there is a light at A and another at B, the one at B having an intensity 8 times that at A. The distance AB is 12 ft. At what point on the line AB will the intensity of illumination be least?

20. A farmer has 100 yd. of fence and wishes to enclose a rectangular plot and divide it into two equal parts by a cross fence joining the mid-

points of the two sides. What dimensions would give the largest possible enclosed area?

21. A cylindrical tank with given volume is to be made without a lid. The material for the bottom costs five times as much per unit area as that for the sides. What should be the ratio of height to radius for minimum cost?

22. A Norman window has the form of a rectangle surmounted by a semicircle. What shape will give the greatest amount of light for a given perimeter of window?

23. What is the shape of the strongest rectangular beam that can be cut from a log of radius r if the strength varies directly as the product of the width and the square of the depth?

24. An open trough is to be made from three boards, each 4 ft. long and 1 ft. wide. One board forms the bottom and the other two form the two sides, the cross section being trapezoidal. What width across the top would give the greatest volume?

25. What is the area of the largest isosceles triangle that can be inscribed in a circle of radius r?

26. A lot has the form of a right triangle with the two legs equal to 60 ft. and 100 ft. A rectangular building is to be erected facing on the longer of these. What are its dimensions for a maximum floor area?

27. One corner of a sheet of width a is turned back so as to touch the opposite edge. Find the minimum length of the crease. See Fig. 9, Chap. I.

28. Find the equation of the ellipse of smallest area which can be circumscribed about a rectangle whose sides are 6 in. and 8 in.*

*The area of an ellipse with semi-axes a and b is πab.

CHAPTER VI

THE SECOND DERIVATIVE

31. Successive differentiation.—The derivative of a function $y = f(x)$ is itself a function of x which can in general be differentiated with respect to x. The result is called the *second derivative* of the original function and is denoted by any of the symbols

$$\frac{d^2y}{dx^2}; \qquad f''(x); \qquad y''.$$

This second derivative may in turn be a differentiable function of x, and its derivative is called the *third derivative* of the original function and so on. The result of differentiating the function $y = f(x)$ n times with respect to x is called the *nth derivative* of the function with respect to x. It is denoted variously by the symbols

$$\frac{d^ny}{dx^n}; \qquad f^{(n)}(x); \qquad y^{(n)}.$$

Example 1

$$y = x^3 - 3x^2 + 6.$$

$$\frac{dy}{dx} \text{ or } y' = 3x^2 - 6x$$

$$\frac{d^2y}{dx^2} \text{ or } y'' = 6x - 6$$

$$\frac{d^3y}{dx^3} \text{ or } y''' = 6.$$

It is obvious that in this case the derivatives of order higher than the third are identically zero.

Example 2

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = +\frac{2}{x^3}$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}.$$

32. Successive differentiation of implicit functions.—If y is defined as a function of x by an implicit equation, the successive derivatives of y with respect to x can be found by the procedure illustrated by the following example:

$$x^2 - y^2 = a^2$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}.$$

Now, the second derivative of y with respect to x is the derivative *with respect to* x of the first derivative; *i.e.*,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= \frac{y - x\dfrac{dy}{dx}}{y^2}.$$

Substituting for dy/dx its value in terms of x and y we have finally

$$\frac{d^2y}{dx^2} = \frac{y - x \cdot \dfrac{x}{y}}{y^2}$$

$$= \frac{y^2 - x^2}{y^3}$$

$$= -\frac{a^2}{y^3}.$$

The third derivative could of course be found now by taking the derivative of this result with respect to x.

PROBLEMS

In each of the following find dy/dx and d^2y/dx^2:

1. $y = 3x^2 + 2x + 12.$

2. $y = x^3 + 7x + 10.$

3. $y = \sqrt{a^2 - x^2}.$

4. $y = \dfrac{b}{a}\sqrt{a^2 + x^2}.$

5. $y = \dfrac{1}{x}.$

6. $y = \dfrac{x}{x - 1}.$

7. $y = \dfrac{x - 1}{x + 1}.$

8. $y = \dfrac{3}{x^2 + x}.$

Find d^2y/dx^2 from each of the following implicit functions:

9. $y^2 = 4px.$

10. $x^2 + y^2 = r^2.$

11. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$

12. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

13. $b^2x^2 + a^2y^2 = a^2b^2.$

14. $xy - x^2 - 4y = 0.$

15. $x^2 + 4y^2 - 2x - 24y + 21 = 0.$

16. $y^2 - 4xy - 2 = 0.$

In each of the following find the derivatives indicated.

17. $xy = 1;\ \dfrac{d^2y}{dx^2},\ \dfrac{d^3y}{dx^3},\ \dfrac{d^ny}{dx^n}.$

18. $y = \dfrac{1}{x^2};\ \dfrac{d^ny}{dx^n}.$

19. $y = x^3 - 7x + 10;\ \dfrac{d^3y}{dx^3},\ \dfrac{d^4y}{dx^4}.$

20. $x^2 + y^2 = 16;\ \dfrac{d^3y}{dx^3}.$

21. $b^2x^2 + a^2y^2 = a^2b^2;\ \dfrac{d^3y}{dx^3}.$

22. What is the value of d^2y/dx^2 at the maximum point on the curve $y = x^3 - 3x^2 + 6$?

23. Show that on the parabola $y = Ax^2 + Bx + C$ the value of d^2y/dx^2 is everywhere positive if A is positive, and everywhere negative if A is negative.

33. Geometrical meaning of the second derivative. Concavity.—Since d^2y/dx^2 is the derivative with respect to x of dy/dx, its value at any point on the curve $y = f(x)$ is the *rate of change of dy/dx*, measured with respect to x. Consequently,

$$\text{Where } \frac{dy}{dx} \text{ is } \textit{increasing}, \ \frac{d^2y}{dx^2} \text{ is } \textit{positive;}$$

$$\text{Where } \frac{dy}{dx} \text{ is } \textit{decreasing}, \ \frac{d^2y}{dx^2} \text{ is } \textit{negative.}$$

Consider, as an example, the graph of

$$y = x^3 - 3x^2 + 6$$

which is shown in Fig. 32. As we move along the curve say from $A(-1, 2)$ to P, the value of dy/dx is continuously *decreasing*—from a large positive value ($+9$) at A, down to 0 at M and on down to a rather large negative value at P. *Throughout this interval the value of d^2y/dx^2 is negative;* the curve is said to be **concave downward** in this interval. The exact location of P is not immediately apparent.

As we continue along the curve from P to say $B(3, 6)$, the value of dy/dx is continuously *increasing*—from its negative value at P up to 0 at N and on up to $+9$ at B. *Throughout this interval the value of d^2y/dx^2 is positive;* the curve is said to be **concave upward** in this interval.

34. Inflection points.—The point P (Fig. 32) is called an *inflection point.* It is a point that separates an arc of the

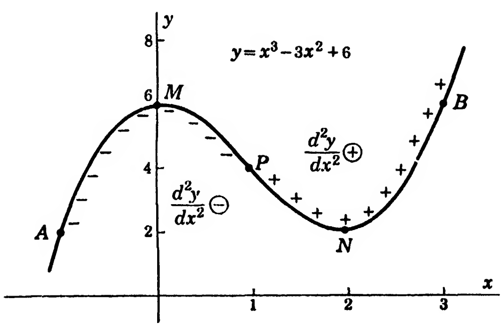

FIG. 32.

curve which is concave upward from one which is concave downward. Such points can usually be found easily; for, since d^2y/dx^2 changes sign when we go through an inflection point, its value must be *zero* at the inflection point if it is continuous. Thus, for the example shown in the figure

we have

$$y = x^3 - 3x^2 + 6$$
$$\frac{dy}{dx} = 3x^2 - 6x$$
$$\frac{d^2y}{dx^2} = 6x - 6.$$

Setting d^2y/dx^2 equal to 0 and solving for x we find

$$6x - 6 = 0$$
$$x = 1.$$

The inflection point is then $P(1, 4)$.

It must be emphasized that the condition which must be satisfied in order that P be an inflection point is that the sign of d^2y/dx^2 must be $+$ on one side and $-$ on the other side of P. The curve $y = x^4$ illustrates the fact that we do not always have an inflection point where $d^2y/dx^2 = 0$. In this case $d^2y/dx^2 = 0$ at $x = 0$, but it is positive for x both to the left and to the right of this point. The curve is everywhere concave upward.

It is also possible to have an inflection point say at $x = a$ without having $d^2y/dx^2 = 0$ at this point; this can happen however only if d^2y/dx^2 "becomes infinite" or is otherwise discontinuous at this point. We shall not consider such cases here.

35. Another method of distinguishing maximum from minimum points.—A glance at Fig. 32 makes it clear that a maximum point on a curve is ordinarily a point at which the tangent line is horizontal *on an arc which is concave downward,* while a minimum point is a point at which the tangent line is horizontal *on an arc which is concave upward.* That is, ordinarily,

At a **maximum** point $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2}$ is **negative.**

At a **minimum** point $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2}$ is **positive.***

* If $d^2y/dx^2 = 0$, this test fails. The test on p. 54 should then be used.

This often affords a more convenient method of distinguishing maximum from minimum points than that used in the last chapter. Thus, in the case of the curve shown in Fig. 32, we have

$$y = x^3 - 3x^2 + 6$$
$$\frac{dy}{dx} = 3x^2 - 6x.$$

Setting $dy/dx = 0$ and solving for x we find that the tangent line is horizontal at

$$x = 0 \qquad \text{and} \qquad x = 2.$$

In order to determine whether these values of x correspond to maximum or minimum points we employ the second derivative as follows:

$$\frac{d^2y}{dx^2} = 6x - 6.$$

$\left.\frac{d^2y}{dx^2}\right]_{x=0}$ is negative; hence this is a maximum point.

$\left.\frac{d^2y}{dx^2}\right]_{x=2}$ is positive; hence this is a minimum point.

The numerical value of the second derivative is of no importance in this particular connection. We are interested only in whether it is positive or negative.

36. Acceleration in rectilinear motion.—We have already found that if a point moves along *a straight line path* so that its distance from a fixed point A on the path varies with the time according to the law

$$s = f(t),$$

the value of ds/dt at any instant is the velocity v of the moving point at that instant.

The rate at which the velocity is changing (with respect to time) is called the *acceleration* of the moving point. This rate is obviously given by the value of

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}.$$

If s is expressed in feet and t in seconds, the velocity is of course in feet per second. The acceleration, being the rate of change of velocity with respect to time, is expressed in feet per second per second; this is often abbreviated by writing ft./sec.^{2} Acceleration may of course also be expressed in such units as miles per hour per hour, miles per hour per second, feet per second per minute, etc.

PROBLEMS

In each of the following, (*a*) determine the maximum and minimum points; (*b*) find the inflection points; (*c*) state in what intervals the curve is concave up and in what intervals it is concave down; (*d*) sketch the curve:

1. $y = x^3 - 3x + 3$.

2. $y = x^3 - 9x^2 + 24x - 20$.

3. $y = x^4 - 16$.

4. $y = x^4 - 32x^2$.

5. $y = \frac{x}{2} + \frac{2}{x}$.

6. $y = \frac{x^3}{3} + \frac{81}{x}$.

7. $y = x^3 + 9x^2 + 27x + 23$.

8. $3y = x^3 + 3x^2 - 9x + 14$.

9. $y = \frac{8a^3}{x^2 + 4a^2}$.

10. $y = \frac{64}{x^2 + 16}$.

11. $32y = x^4 - 8x^3 - 8x^2 + 96x + 16$.

12. $y = 3x^4 - 8x^3 + 2$.

13. A man driving an automobile notices at a certain instant that his speed is 40 m.p.h.; 3 min. later the speed is 58 m.p.h. What is his average acceleration?

14. If a small heavy object is thrown vertically upward from the ground with initial velocity v_0, its distance above the ground after t seconds is given by

$$s = v_0t - \tfrac{1}{2}gt^2.$$

Show that its acceleration is constant.

15. A particle moves along a line so that its distance from a fixed point A on the line is given by

$$s = t^2 + \frac{2}{t}.$$

Find its velocity and acceleration at the end of 2 sec.

16. A point moves along the x-axis so that its distance from the origin at the end of t sec. is

$$s = \sqrt{at + b}.$$

Show that its acceleration is negative and inversely proportional to the cube of s. Describe the motion.

CHAPTER VII

THE TRIGONOMETRIC FUNCTIONS

37. Review of definitions and fundamental relations.—Before entering into a study of the derivatives of the trigonometric functions it is necessary to review briefly the definitions of these functions and their fundamental properties.

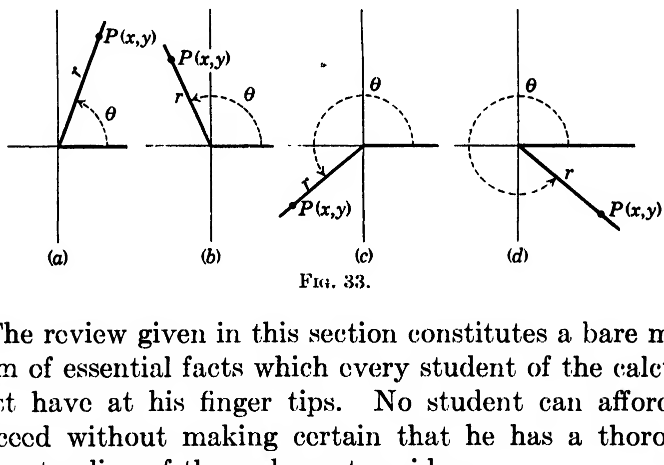

Fig. 33.

The review given in this section constitutes a bare minimum of essential facts which every student of the calculus must have at his finger tips. No student can afford to proceed without making certain that he has a thorough understanding of these elementary ideas.

1. *The definitions.*—If θ is any *number*, the trigonometric functions of θ are defined as follows: Construct an angle of θ radians with vertex at the origin and initial side along positive x-axis, measuring the angle counterclockwise if θ is positive. Choose any point $P(x, y)$ on the terminal side; denote its distance from the origin by r. Then (Fig. 33)

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}.$$

2. *The signs of the functions.*—We may agree to consider that: x is positive when measured to the right and negative to the left of the origin; y is positive when measured upward and negative downward from the origin. We may further agree to regard r as always positive.

With these conventions as to the signs of x, y, and r in mind, one can easily determine whether a given function of a given angle is positive or negative. Thus if α is an angle whose terminal side lies in the second quadrant, $\cos\alpha$ is negative; for $\cos\alpha = x/r$ and x is $-$ while r is $+$.

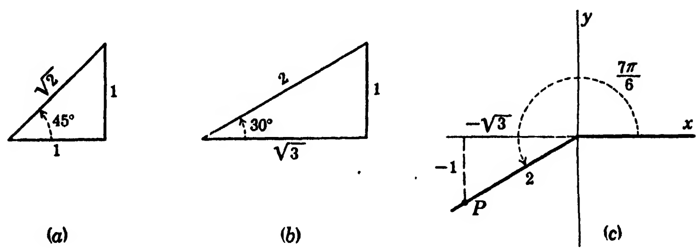

FIG. 34.

3. *Values of the functions for certain angles.*—In order to find the value of any function of any angle which is a multiple of 30 or 45°, one has only to remember that in the 45° right triangle the two legs are equal and in the 30–60° right triangle the shortest leg is exactly equal to one-half of the hypotenuse. The values of x, y, and r may then be taken as shown in Fig. 34, *a* or *b*. Thus, to find sin 210° or, as we prefer, $\sin 7\pi/6$, one draws Fig. 34*c* and writes down from it

$$\sin\frac{7\pi}{6} = -\tfrac{1}{2}.$$

The values of the functions for the quadrantal angles 0, $\pi/2$, π, and $3\pi/2$ can also be easily found. For this, one needs only to note that for such angles either x or y is zero

and the other equals $\pm r$. Thus for the angle $3\pi/2$, $x = 0$ and $y = -r$; hence (Fig. 35),

$$\sin \frac{3\pi}{2} = \frac{-r}{r} = -1.$$

$$\cos \frac{3\pi}{2} = \frac{0}{r} = 0.$$

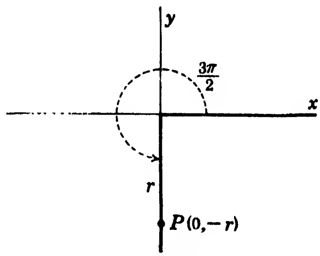

FIG. 35.

4. *The values of the functions when that of one is known.*—If, for a certain angle, the value of one of the six trigonometric functions and the quadrant in which the terminal side lies are known, the values of the other five functions can be found easily. Several problems involving this are given in the next set.

5. *The fundamental identities.*—From the way in which the trigonometric functions of θ are defined it is evident that they are not independent of each other. The student should memorize and be able to prove the following fundamental relations which exist between them:

(a) $\sin^2 \theta + \cos^2 \theta = 1.$

(b) $1 + \tan^2 \theta = \sec^2 \theta.$

(c) $1 + \cot^2 \theta = \csc^2 \theta.$

(d) $\tan \theta = \dfrac{\sin \theta}{\cos \theta}.$

(e) $\begin{cases} \csc \theta = \dfrac{1}{\sin \theta}. \\ \sec \theta = \dfrac{1}{\cos \theta}. \\ \cot \theta = \dfrac{1}{\tan \theta}. \end{cases}$

The proofs follow immediately from the definitions of the functions.

6. *The functions of $-\theta$.*—By comparing the values of x, y, and r for any angle θ with those for $-\theta$ (an angle equal to θ but measured *clockwise* from the positive x-axis), it is easy to show that

$$\sin(-\theta) = -\sin \theta;$$
$$\cos(-\theta) = \cos \theta;$$
$$\tan(-\theta) = -\tan \theta.$$

7. *Reduction to acute angles.*—Tables ordinarily give the values of the functions for angles only up to 90°. For a

larger angle the value of any function can be found as follows:

Subtract from the angle whatever multiple of 90° is necessary in order to have a remainder less than 90°. Then take the same function or the cofunction of the remainder according as an even or odd multiple of 90° was subtracted. That is

$$\textbf{Any function of } (n \cdot 90° + \theta) = \pm \begin{cases} \textbf{same function of } \theta \textbf{ if } n \textbf{ is even.} \\ \textbf{cofunction of } \theta \textbf{ if } n \textbf{ is odd.} \end{cases}$$

The sign is determined by the particular function required and the quadrant in which the terminal side lies, as discussed in Part 2 of this section.

Example

Find sin 285° from the tables.

Solution

$$\sin 285° = \sin (3 \cdot 90° + 15°) = \pm \cos 15°.$$

Since the terminal side of 285° lies in the fourth quadrant, sin 285° must be *negative;* hence,

$$\sin 285° = -\cos 15° = -.9659.$$

8. *The functions of the sum and difference of two angles. Double- and half-angle formulas.*—From the definitions of the functions, it can be shown that if one adds together two angles x and y, the sine, cosine, and tangent of the sum are

(a) $\sin (x + y) = \sin x \cos y + \cos x \sin y.$

(b) $\cos (x + y) = \cos x \cos y - \sin x \sin y.$

(c) $\tan (x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$

Replacing y by $-y$ in (a), (b), and (c), we have for the difference between two angles,

(d) $\sin (x - y) = \sin x \cos y - \cos x \sin y.$

(e) $\cos (x - y) = \cos x \cos y + \sin x \sin y.$

(f) $\tan (x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$

If $y = x$ formulas (a), (b), and (c) reduce to

(g) $$\sin 2x = 2 \sin x \cos x.$$
(h) $$\cos 2x = \cos^2 x - \sin^2 x.$$
(i) $$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

From formula (h) one can easily show that

(j) $$\sin \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{2}}.$$

(k) $$\cos \tfrac{1}{2}x = \pm\sqrt{\frac{1 + \cos x}{2}}.$$

(l) $$\tan \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}.$$

It should be noticed that the $\pm$ sign is omitted in the last two forms of the formula for $\tan \frac{1}{2}x$. This is because $(1 \pm \cos x)$ is never negative and $\tan \frac{1}{2}x$ always has the same sign as $\sin x$.

9. *Sine and cosine laws.*—**Sine law:** In any triangle the sides are proportional to the sines of the opposite angles. That is (Fig. 36),

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

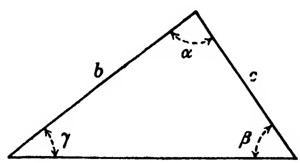

FIG. 36.

Cosine law: The square of any side of a triangle is equal to the sum of the squares of the two other sides minus twice the product of these sides and the cosine of the included angle. Thus, in Fig. 36,

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

10. *General behavior and graphs of the functions.*—The functions $\sin x$ and $\cos x$ are single-valued and continuous for all values of x. The same statement applies to $\tan x$ except at the points where x is an odd multiple of $\pi/2$; at these points it is undefined.

The manner in which each of these functions varies with x can be discussed in a general way from the corresponding graph. Thus, Fig. 37 indicates that $\sin x$ increases from 0 at $x = 0$ to 1 at $x = \frac{1}{2}\pi$, the *rate* of increase becoming continuously smaller as we move from left to right. In the

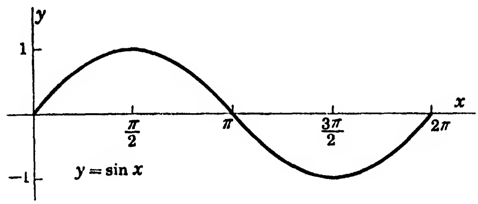

FIG. 37.

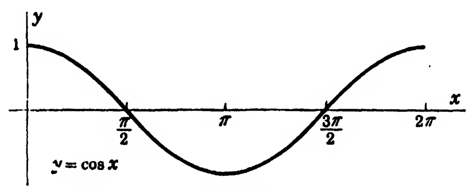

FIG. 38.

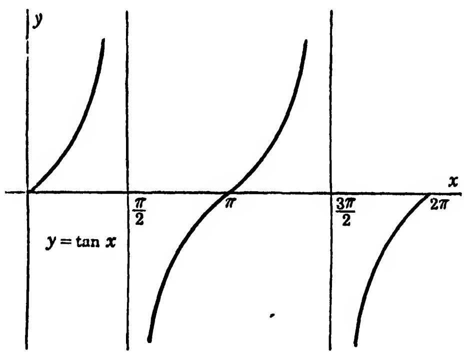

FIG. 39.

interval from $x = \frac{1}{2}\pi$ to $x = \frac{3}{2}\pi$, $\sin x$ decreases from $+1$ to -1; it then increases to 0 at $x = 2\pi$. The function is periodic with period 2π; *i.e.*, for any value of x,

$$\sin (x + 2\pi) = \sin x.$$

A similar discussion could be given for the functions $\cos x$ and $\tan x$ whose graphs are shown in Figs. 38 and 39.

Since cot x, sec x, and csc x are the reciprocals of these functions, their graphs may easily be sketched and their general behavior discussed from the graphs.

PROBLEMS

Evaluate the following expressions without using tables:

1. $\sin 135° + \tan 210° - \cos 315° + \cot 60°$.

2. $\sec 0 - \cos \pi + 2 \sin \frac{3\pi}{2} - \cos \frac{\pi}{2}$.

3. $3 \sec \pi + 4 \cos \frac{\pi}{3} - \csc \frac{5\pi}{3} + \tan 0$.

4. $2 \sin \frac{13\pi}{6} - \cos 660° + 2 \tan 495°$.

5. Using tables find $\sin 700°$, $\cos(-820°)$, $\tan 1000°$.

6. From a figure prove that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ if the terminal side of x lies in the first quadrant. The third quadrant.

7. Prove the fundamental identities listed in Part 5, of this section.

8. Prove the formulas for $\sin(x + y)$ and $\cos(x + y)$ assuming that $(x + y) < \frac{1}{2}\pi$. How may the proof be extended to cover all values of x and y?

9. Show how all of the rest of the formulas concerning the functions of the sum and difference of two angles, and the double- and half-angle formulas, are derived from those for $\sin(x + y)$ and $\cos(x + y)$.

10. If $\tan x = \frac{3}{4}$ and $x < \pi/2$, find $\sin 2x$, $\cos 2x$, and $\tan \frac{1}{2}x$.

11. If $\tan x = -\frac{5}{12}$ and the terminal side of x lies in the second quadrant, find $\sin 2x$, $\cos 2x$, and $\tan \frac{1}{2}x$.

12. If $\cos x = \frac{5}{13}$ and $\sin y = \frac{3}{5}$, the terminal side of x being in the fourth quadrant and that of y in the second, find $\sin(x + y)$ and $\tan(x - y)$.

13. Prove the sine law and the cosine law.

14. Sketch the curve $y = \sin x$ for $x = 0$ to $x = 2\pi$; then, on the same axes, sketch $y = \csc x$ using the fact that $\csc x = 1/\sin x$. Discuss the way in which $\csc x$ varies with x.

15. Solve Prob. 14 for the curves $y = \cos x$ and $y = \sec x$. For $y = \tan x$ and $y = \cot x$.

16. Sketch on the same axes the curves $y = 2 \sin x$ and $y = \sin 2x$. How do their periods and amplitudes compare?

17. Show that the function $a \sin bx$ is periodic with period $2\pi/b$. What is its amplitude? How may the graph be sketched easily?

18. Sketch the curve $y = 3 \sin 4x + 2 \cos 2x$ by adding ordinates.

19. Solve the equation: $2 \sin^2 \theta - \cos \theta - 1 = 0$.

20. Solve the equation: $4 \tan^2 \theta - 3 \sec \theta = 6$.

21. Solve the equation: $2(1 - \cos 2x) = \sin x + \cos 2x$.

22. Show that $\sin 2\theta = \dfrac{2 \tan \theta}{1 + \tan^2 \theta}$.

23. Simplify: $2 \sin^3 x \cos x + 2 \cos^3 x \sin x$.

24. Simplify:

$$\frac{\left[1 + \left(\dfrac{\sin \theta}{1 - \cos \theta}\right)^2\right]^{\frac{3}{2}}}{\dfrac{1}{(1 - \cos \theta)^2}}.$$

25. The following formulas express the sum or difference of sines or cosines of two angles as a product. Show how they are derived from the formulas for the sine and cosine of $(x + y)$ and $(x - y)$.

(*a*) $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$.

(*b*) $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

(*c*) $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$.

(*d*) $\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$.

38. The derivative of sin v.—To find the derivative of the function $y = \sin x$ we must apply the fundamental process of differentiation. Starting at any point $P(x, \sin x)$ on the curve, and letting x increase by a small amount Δx we have (Fig. 40)

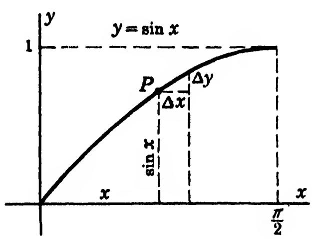

FIG. 40.

$$\Delta y = \sin (x + \Delta x) - \sin x;$$
$$\frac{\Delta y}{\Delta x} = \frac{\sin (x + \Delta x) - \sin x}{\Delta x}.$$

The required derivative is of course the limit of this quotient as $\Delta x \to 0$. This limit is not apparent from the present form of the fraction; however, if we apply the formula

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$$

to the numerator, the fraction becomes

$$\frac{\Delta y}{\Delta x} = \frac{2 \cos (x + \frac{1}{2}\Delta x) \sin \frac{1}{2}\Delta x}{\Delta x}$$
$$= \cos (x + \tfrac{1}{2}\Delta x) \frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}.$$

We have then immediately,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} (\cos x + \tfrac{1}{2}\Delta x) \lim_{\Delta x \to 0} \frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}$$
$$= \cos x. \quad \text{Explain.}$$

If we consider the more general function $y = \sin v$ where v is a differentiable function of x, the derivative *with respect to* v is $\cos v$ by virtue of the above result. To obtain the derivative with respect to x we must multiply this by dv/dx. That is

$$\text{(IX)} \qquad \frac{d}{dx}\sin v = \cos v \frac{dv}{dx}.$$

Thus, for the function

$$y = \sin 4x,$$
$$\frac{dy}{dx} = \cos 4x \frac{d}{dx}(4x)$$
$$= 4 \cos 4x.$$

39. Derivatives of the other trigonometric functions.—The derivatives of the other trigonometric functions could of course be obtained by the procedure used above. They can be found more easily, however, by other means. Thus, since

$$\cos v = \sin\left(\frac{\pi}{2} - v\right),$$

we have

$$\frac{d}{dx}\cos v = \frac{d}{dx}\sin\left(\frac{\pi}{2} - v\right)$$
$$= \cos\left(\frac{\pi}{2} - v\right)\frac{d}{dx}\left(\frac{\pi}{2} - v\right) \quad \text{[by (IX)]}$$
$$= -\sin v \frac{dv}{dx}.$$

The formula for the derivative of $\tan v$ can now be obtained by writing

$$\tan v = \frac{\sin v}{\cos v}$$

and differentiating the quotient. The formulas are all listed below; the proofs are left to the exercises.

(IX) $$\frac{d}{dx}\sin v = \cos v\frac{dv}{dx}.$$

(X) $$\frac{d}{dx}\cos v = -\sin v\frac{dv}{dx}.$$

(XI) $$\frac{d}{dx}\tan v = \sec^2 v\frac{dv}{dx}.$$

(XII) $$\frac{d}{dx}\cot v = -\csc^2 v\frac{dv}{dx}.$$

(XIII) $$\frac{d}{dx}\sec v = \sec v\tan v\frac{dv}{dx}.$$

(XIV) $$\frac{d}{dx}\csc v = -\csc v\cot v\frac{dv}{dx}.$$

In finding the derivative of a function that involves the trigonometric functions one must use formulas (IX) to (XIV) in addition to formulas (I) to (VIII). The details are illustrated by the following examples:

Example 1

$$y = \sqrt{\tan x}.$$

$$\frac{dy}{dx} = \tfrac{1}{2}(\tan x)^{-\frac{1}{2}}\frac{d}{dx}\tan x \qquad \text{[by (V)]}$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}. \qquad \text{[by (XI)]}$$

Example 2

$$y = \sec^2 2\theta.$$

$$\frac{dy}{d\theta} = 2\sec 2\theta\frac{d}{d\theta}\sec 2\theta \qquad \text{[by (V)]}$$

$$= 2\sec 2\theta\sec 2\theta\tan 2\theta\frac{d}{d\theta}2\theta \qquad \text{[by (XIII)]}$$

$$= 4\sec^2 2\theta\tan 2\theta.$$

Example 3

$$y = \frac{\sin x}{x}.$$

$$\frac{dy}{dx} = \frac{x\dfrac{d}{dx}\sin x - \sin x\dfrac{dx}{dx}}{x^2} \qquad \text{[by (VI)]}$$

$$= \frac{x\cos x - \sin x}{x^2}.$$

PROBLEMS

1. Derive the formula for the derivative of $\cos x$ using the fundamental differentiation process.

2. Derive the formula for the derivative of $\tan x$ using the fundamental differentiation process.

3. Derive formulas (XI) to (XIV) inclusive without using the fundamental process.

Differentiate the following functions:

4. $y = 4 \sin 3x.$

5. $y = 2x + \cos x.$

6. $y = \sin \theta \cos^2 \theta.$

7. $y = 3 \cot^2 x.$

8. $y = \sin^2 \frac{1}{2}\theta.$

9. $y = \sqrt{\sin 2x}.$

10. $y = \sin^2 x \sec x.$

11. $s = t \cos t.$

12. $y = x^2 \tan^2 x.$

13. $y = 2a \csc^2 \frac{1}{2}x.$

14. $y = x^{\frac{1}{2}} \sec x.$

15. $y = \dfrac{\cos x}{x}.$

16. $y = \dfrac{\sin x}{1 - \cos x}.$

17. $y = \dfrac{\sec x + \tan x}{\sec x - \tan x}.$

18. $y = \dfrac{1 - \sin x}{1 + \sin x}.$

19. $y = \dfrac{1 - \tan^2 x}{\tan x}.$

20. $y = 4 \sin^2 (x^2 + 6).$

21. $y = \sqrt{x} \cos (1 - x^2).$

In each of the following, find d^2y/dx^2:

22. $y = \sin^2 x.$

23. $y = \sec \frac{1}{2}x.$

24. $y = \sin^2 x \cos x.$

25. $y = \tan^2 \frac{1}{2}x.$

26. $y = \dfrac{\sin x}{1 - \cos x}.$

27. $y = \dfrac{\sin x}{x}.$

28. Show that at any point on the curve $y = \sin x$ the value of d^2y/dx^2 is numerically equal to that of y.

29. Prove that the curves $y = \sin x$ and $y = \tan x$ have the same tangent line at the origin and that its inclination is 45°.

30. At what angle do the curves $y = \sin x$ and $y = \cos x$ intersect?

31. At what angles do the curves $y = \sin x$ and $y = \frac{1}{2} \tan x$ intersect?

32. Show that when $x = \pi/4$, $\tan x$ is increasing at a rate of 2 units per unit of increase in x.

33. Show that the derivative of $\tan x$ with respect to $\sin x$ is $\sec^3 x$; hence show that when $x = \pi/3$, $\tan x$ is increasing 8 times as fast as $\sin x$.

For each of the following curves, locate the maximum, minimum, and inflection points. Sketch the curves:

34. $y = \sin^2 x$.
35. $y = \sin^4 x$.
36. $y = \sin x + \cos x$.
37. $y = 4 \sin x + 3 \cos x$.

38. $y = 3 \tan x - 4x$. $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$.

39. $y = 4 \sin^2 x + 3 \cos 2x$.

40. A man starts at a point A and walks 40 ft. north, then turns and walks east at 5 ft. per second. If a searchlight placed at A follows him, at what rate is it turning at the end of 6 sec.?

41. A triangle has two equal sides of length L. What is the angle between these sides if the area of the triangle is a maximum?

42. On the top of a wall 27 ft. high is another wall which is inset 1 ft. What are the length and the inclination of the shortest ladder that would reach from the ground to this wall?

43. A corridor $13\frac{1}{2}$ ft. wide meets at right angles another which is 4 ft. wide. Could a pole 24 ft. long be carried horizontally around the corner?

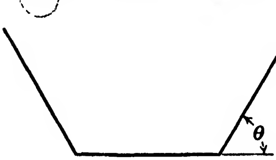

FIG. 41.

44. An open gutter with sloping sides of equal inclination is to be made from a long piece of sheet metal which is 15 in. wide, by bending up one-third of the sheet on each side as indicated in Fig. 41. For what inclination θ of the sides is the capacity a maximum?

45. A right circular cone is inscribed in a sphere of radius a. Find its height if the lateral surface is to be a maximum. HINT: Let the semiangle θ at the vertex be the independent variable. Note from Fig. 42 that the length of an element is $2a \cos \theta$; hence $h = 2a \cos^2 \theta$ and $r = 2a \cos \theta \sin \theta$.

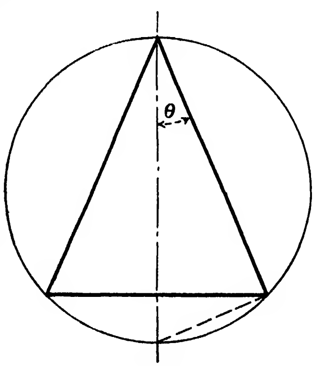

FIG. 42.

46. Solve Prob. 45 if the volume instead of lateral surface of the cone is to be a maximum.

47. A right circular cone is circumscribed about a right circular cylinder of radius r and height h. Show that the volume of the cone is a minimum if $\tan \theta = r/2h$, where θ is the semiangle at the vertex.

48. A right circular cone is placed inside a sphere of radius a with its vertex at the center. For what semiangle θ at the vertex is the volume of the cone a maximum?

49. A right circular cone is circumscribed about a sphere of radius a. If θ is the semiangle at the vertex, show that the volume of the cone will be a minimum if $\sin\theta = \frac{1}{3}$.

50. A particle moves around the circle $x^2 + y^2 = r^2$ at a constant angular velocity of ω radians per second. Show that the projection of the particle on the x-axis moves back and forth from $(-r, 0)$ to $(+r, 0)$ in such a way that its acceleration is always toward the origin and proportional to its distance from the origin. This type of motion of a point along a straight line is called *simple harmonic motion.*

40. The functions arc sin x and arc cos x.—We define the abbreviation

$$y = \text{arc} \sin x, \qquad \text{or} \qquad y = \sin^{-1} x^*$$

to mean "y is the radian measure of an angle whose sine is x." For each value of x between -1 and $+1$ this relation of course yields an indefinite number of values of y. Thus, corresponding to $x = \frac{1}{2}$ we have

$$y = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots, -\frac{7\pi}{6}, -\frac{11\pi}{6}, \ldots.$$

The function is undefined for values of x greater than $+1$ or less than -1.

If we solve the equation for x in terms of y we obtain

$$x = \sin y;$$

hence the graph of $y = \text{arc} \sin x$ is the same as that of $y = \sin x$ with the axes interchanged. See Fig. 43. The function can be made single-valued by agreeing to use only that part of the curve from A to B. This part is called the *principal branch* of the curve, and the corresponding values of the function are called the *principal values.* Throughout this work we shall assume

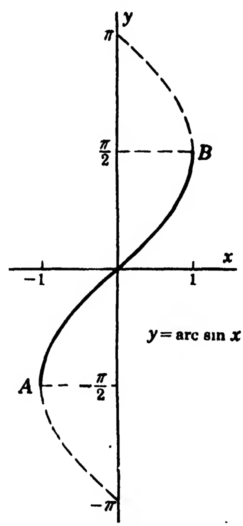

FIG. 43.

* Since it appears that neither notation will be universally adopted in the near future, the student must be familiar with both. It should be emphasized that the index -1 in the symbol $\sin^{-1} x$ is *not* an exponent, but is an integral part of the symbol.

this restriction to be made; thus, for example, we agree that

$$\text{arc sin } \tfrac{1}{2} = \frac{\pi}{6}; \qquad \text{arc sin } -\tfrac{1}{2} = -\frac{\pi}{6}.$$

The function $y = \text{arc sin } x$ thus becomes a single-valued continuous function, defined for all values of x in the interval $-1 \leqq x \leqq +1$ and having for its graph the arc AB in Fig. 43. It is obvious from the graph that the function is everywhere increasing; *i.e., its derivative is everywhere positive.*

To obtain its derivative we observe that, if

$$y = \text{arc sin } x,$$

then

$$x = \sin y,$$

and

$$\frac{dx}{dy} = \cos y.$$

Then, using formula (VIII),

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

But

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2};$$

hence,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

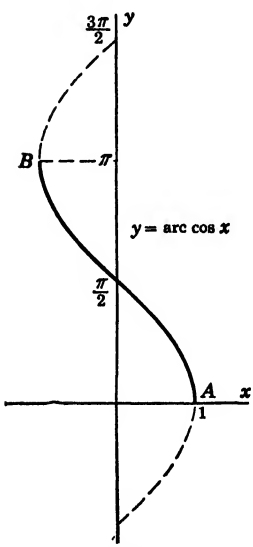

FIG. 44.

It should be noticed that the positive sign only is used here for the radical. This is because we are considering only the principal branch of the function.

In a similar manner we define the function

$$y = \text{arc cos } x$$

as a single-valued continuous function in the interval from $x = -1$ to $x = +1$, having for its graph the arc AB in

Fig. 44. Considering only this principal branch we have, for example,

$$\text{arc cos } \tfrac{1}{2} = \frac{\pi}{3}; \qquad \text{arc cos } -\tfrac{1}{2} = \frac{2\pi}{3}.$$

The derivative of this function is obviously everywhere negative. To obtain its derivative we proceed exactly as in the previous case and find that for

$$y = \text{arc cos } x,$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

Finally, if v is a differentiable function of x, we have the formulas

$$\text{(XV)} \qquad \frac{d}{dx}(\text{arc sin } v) = \frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}.$$

$$\text{(XVI)} \qquad \frac{d}{dx}(\text{arc cos } v) = -\frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}.$$

Example

$$y = \text{arc sin } 2x.$$
$$\frac{dy}{dx} = \frac{\frac{d}{dx}(2x)}{\sqrt{1 - (2x)^2}}$$
$$= \frac{2}{\sqrt{1 - 4x^2}}.$$

41. The functions arc tan x and arc cot x.—The relation

$$y = \text{arc tan } x, \qquad \text{or} \qquad y = \tan^{-1} x$$

means that y is the radian measure of an angle whose tangent is x. The function is defined for all values of x and is of course multiple-valued. It can be made single-valued by agreeing to use only the part indicated by AB in Fig. 45. Thus, considering only this *principal branch* of the function we have, for example,

$$\text{arc tan } 1 = \frac{\pi}{4}; \qquad \text{arc tan } -1 = -\frac{\pi}{4}.$$

It is obvious from the graph that the derivative of this function is everywhere positive. To find its value we

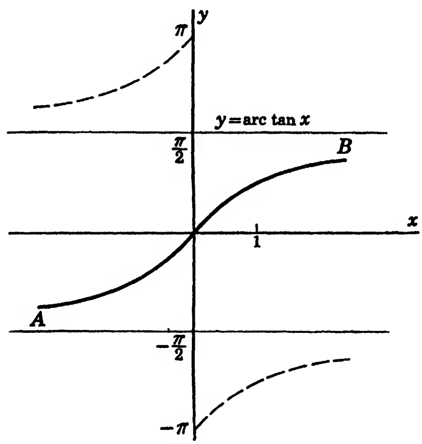

FIG. 45.

proceed as before, noting that, if

$$y = \text{arc tan } x,$$

then

$$x = \tan y$$

and

$$\frac{dx}{dy} = \sec^2 y.$$

Then, using formula (VIII),

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

In exactly the same manner we define the function

$$y = \text{arc cot } x$$

as a single-valued continuous function for all values of x by restricting ourselves to the branch indicated by AB in Fig. 46. It is evident that the derivative of this function

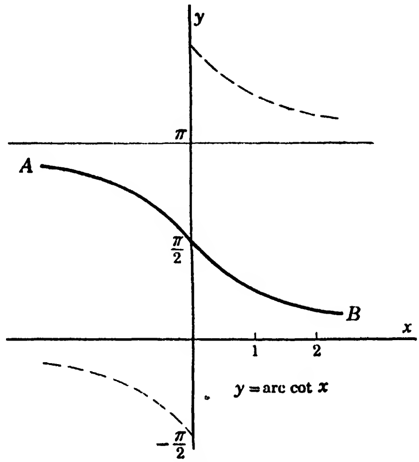

FIG. 46.

is everywhere negative, and it is easy to show that in this case

$$\frac{dy}{dx} = -\frac{1}{1 + x^2}.$$

Finally, if v is a differentiable function of x we have the formulas,

$$\text{(XVII)} \qquad \frac{d}{dx}\operatorname{arc\,tan} v = \frac{\frac{dv}{dx}}{1 + v^2}.$$

$$\text{(XVIII)} \qquad \frac{d}{dx}\operatorname{arc\,cot} v = -\frac{\frac{dv}{dx}}{1 + v^2}.$$

Example **1**

$$y = \operatorname{arc\,tan}\frac{2x - 1}{3}.$$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\frac{2x-1}{3}\right)}{1+\left(\frac{2x-1}{3}\right)^2} \qquad \text{[by (XVII)]}$$

$$= \frac{\frac{2}{3}}{1+\frac{(2x-1)^2}{9}}$$

$$= \frac{6}{9+(2x-1)^2}.$$

Example **2**

$$y = x^2 \text{ arc cot } \tfrac{1}{2}x.$$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\text{arc cot } \tfrac{1}{2}x) + \text{arc cot } \tfrac{1}{2}x\frac{d}{dx}(x^2) \qquad \text{[by (IV)]}$$

$$= x^2\left[-\frac{\frac{d}{dx}(\frac{1}{2}x)}{1+(\frac{1}{2}x)^2}\right] + 2x \text{ arc cot } \tfrac{1}{2}x$$

$$= x^2\left[-\frac{\frac{1}{2}}{1+(\frac{1}{2}x)^2}\right] + 2x \text{ arc cot } \tfrac{1}{2}x$$

$$= -\frac{2x^2}{4+x^2} + 2x \text{ arc cot } \tfrac{1}{2}x.$$

PROBLEMS

1. What change must be made in the formulas for the derivatives of arc sin x and arc cos x if the restriction to the principal branch is removed? Answer the same question for arc tan x and arc cot x.

2. Show that the slope of y = arc tan x is never more than 1 while that of y = arc sin x is never less than 1.

3. Sketch the curve y = arc sec x and derive an expression for its derivative. Why is it desirable to take the part from 0 to $\pi/2$ and $-\pi/2$ to $-\pi$ as the principal branch instead of the part from 0 to π?

4. Solve Prob. 3 for the function y = arc csc x.

Differentiate the following functions:

5. $y = \text{arc cos } 4x.$ **6.** $y = \text{arc sin } \tfrac{1}{2}x.$

7. $y = \text{arc sin } \sqrt{x}.$ **8.** $y = \text{arc cos } \dfrac{1}{x}.$

9. $y = \text{arc cos } \dfrac{1}{\sqrt{1+x^2}}.$ **10.** $y = 4 \text{ arc sin } x^2.$

11. $y = \text{arc tan } 3x.$ **12.** $y = \text{arc cot } \tfrac{1}{2}\theta.$

13. $y = \text{arc tan } \dfrac{x}{\sqrt{1-x^2}}.$ **14.** $y = \text{arc cot } \dfrac{1}{x}.$

15. $y = \text{arc sin } \dfrac{1-x}{1+x}.$ **16.** $y = x^2 \text{ arc sin } x.$

17. $y = x \text{ arc tan } x^2$.

18. $y = x \text{ arc sec } \frac{1}{x}$.

In each of the following, find d^2y/dx^2:

19. $y = \text{arc sin } \frac{1}{2}x$.

20. $y = 4 \text{ arc cos } 3x$.

21. $y = \text{arc tan } 2x$.

22. $y = \text{arc cot } \frac{1}{2}x$.

23. $y = \text{arc sin } \frac{1}{\sqrt{1 + x^2}}$.

24. $y = \text{arc cot } \frac{x}{\sqrt{1 - x^2}}$.

25. Find the inflection point on the curve $y = \text{arc tan } x$.

26. Show that the part of $y = \text{arc cos } x$ for which $x < 0$ is concave upward while the part for which $x > 0$ is concave downward.

27. Show that arc sin x increases at the same rate as x when $x = 0$ and more rapidly than x at every other point.

28. The radius of a right circular cone is 2 ft. If the altitude is 4 ft. and is increasing at 3 in. per minute, at what rate is the vertex angle changing?

29. A kite is 90 ft. above the ground with 150 ft. of string out. It is moving horizontally, directly away from the boy who holds the string, at a speed of 15 ft. per second. At what rate is the inclination of the string to the horizontal changing?

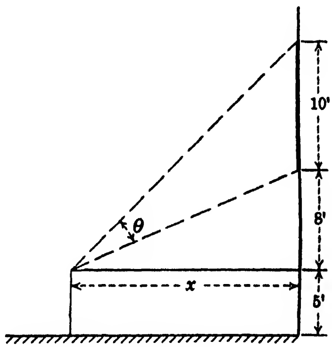

FIG. 47.

30. Find the equation of the line which is tangent to the curve $y = \text{arc tan } x$ at the point where $x = 1$.

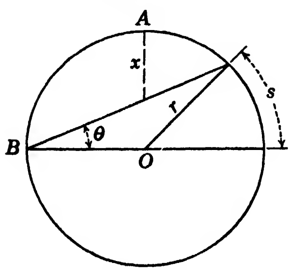

FIG. 48.

31. Sketch on the same axes the curves $y = \text{arc tan } x$ and $y = \text{arc tan } 2x$. Find their angle of intersection.

32. A sign 10 ft. high is erected with its lower edge 13 ft. above the ground. At what distance would a man whose eyes are 5 ft. above the ground obtain the best view, assuming that this is when the angle subtended by the sign at the eye is a maximum? See Fig. 47.

33. A circular race track is surrounded by a board fence. A man walks from A (Fig. 48) toward the center O at 5 ft. per second. If there is a light at B, at what rate will his shadow be moving along the fence when he is two-thirds of the way from A to O? HINT: Note that $s = r \cdot 2\theta$ where $\theta = \text{arc tan } \frac{r - x}{r}$.

CHAPTER VIII

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

42. Review.—At first we define the symbol a^n for n a *positive integer* only, as follows:

$$\begin{aligned} a^1 &= a \\ a^2 &= a \cdot a \\ a^3 &= a \cdot a \cdot a \\ a^n &= a \cdot a \cdot a \cdots \text{to } n \text{ factors.} \end{aligned}$$

This definition obviously assigns no meaning whatever to a^x if x is not a positive integer.

From the above definition one can easily deduce the following laws which govern the use of exponents:

$$(1) \qquad a^m \cdot a^n = a^{m+n}.$$

$$(2) \qquad \frac{a^m}{a^n} = a^{m-n} \text{ if } m > n, \qquad (a \neq 0)$$

$$(3) \qquad (a^m)^n = a^{mn}.$$

$$(4) \qquad (a \cdot b)^m = a^m b^m.$$

$$(5) \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \qquad (b \neq 0)$$

We wish next to extend our definition in order to give a meaning to such symbols as a^0, a^{-2} and $a^{\frac{1}{2}}$. Our guiding principle in defining these symbols is that *the rules which govern the use of positive integral exponents shall apply in all cases.* Thus, if we wish that

$$a^m \cdot a^0 = a^{m+0} = a^m, \qquad (a \neq 0)$$

we must assign the value *one* to the symbol a^0 if $a \neq 0$. We do not here assign any meaning to the symbol 0^0.

Considering negative exponents next, we wish that

$$a^m \cdot a^{-m} = a^{m-m} = a^0 = 1, \qquad (a \neq 0)$$

Hence, we must agree that $a^{-m} = 1/a^m$.

Similarly, for fractional exponents, if we wish that

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a, \qquad (a > 0),$$

we must define the symbol $a^{\frac{1}{2}}$ to stand for a square root of a. To avoid ambiguity we may define it to stand for the *positive* square root. In general we define the symbol $a^{\frac{1}{q}}$ to stand for the positive qth root of a.* Then of course $a^{\frac{p}{q}}$ stands for the pth power of this root.

We thus assign a definite value to a^x $(a > 0)$ for all *rational* values of x. We shall assume, without attempting to justify the assumption here, that there is also a definite value of a^x if x is irrational. Thus, for example, $5^{\sqrt{2}}$ represents a definite number between $5^{1.41}$ and $5^{1.42}$.

43. The exponential function $y = a^x$, $(a > 1)$.—The definitions just discussed, together with the assumption concerning irrational exponents, give a definite value to a^x for every real value of x; *i.e.*, the function

$$y = a^x$$

is a single-valued function, defined for all values of x. It is a continuous function and its graph (if $a > 1$) has the general form shown in Fig. 49. It is evident from the graph that its derivative is everywhere positive and increases with increasing values of x. We shall shortly attempt to compute its derivative.

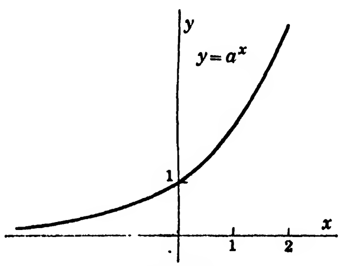

FIG. 49.

44. The logarithmic function $y = \log_a x$, $(a > 1)$.—The logarithmic function is defined as the inverse of the expo-

* If $a > 0$. For $a < 0$, $a^{\frac{1}{q}}$ is undefined in the domain of real numbers if q is even, and is negative if q is odd.

nential function; *i.e.*, if

$$a^y = x,$$

then

$$y = \log_a x.$$

It is evident that the graph of this function (Fig. 50) is the same as that of $y = a^x$ with the axes interchanged. The function is defined only for positive values of x,* is single-valued and continuous, and is everywhere increasing; *i.e.*, its derivative is everywhere positive. Before trying to compute its derivative, we must next introduce an important limit which will arise in that process.

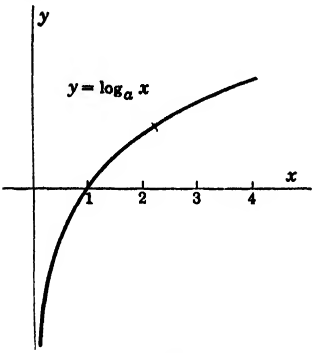

FIG. 50.

45. The number *e*.—Consider the function

$$(1 + v)^{\frac{1}{v}}$$

and think of v becoming smaller and smaller (in absolute value) approaching zero. Evidently the exponent becomes numerically larger and larger while simultaneously the quantity $(1 + v)$ approaches 1. What happens to the value of the function? The following table gives a partial answer.

Value of v....................	0.5	0.1	0.01	0.001
Value of $(1 + v)^{\frac{1}{v}}$.............	2.25	2.5937	2.7048	2.7169
Value of v....................	-0.5	-0.1	-0.01	-0.001
Value of $(1 + v)^{\frac{1}{v}}$.............	4.00	2.8680	2.7320	2.7196

It appears probable from this table that, although the function $(1 + v)^{\frac{1}{v}}$ has no value when v *equals* 0, its limit as

* In the domain of real numbers.

v approaches 0 may exist; *i.e.*, there probably is some constant in the neighborhood of 2.7 to which the values of $(1 + v)^{\frac{1}{v}}$ are arbitrarily near for all values of v which are sufficiently near to 0 but $\neq 0$.

Assuming this limit to exist, we may try to compute it as follows: Let v approach zero by taking the values indicated by the sequence,

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \cdots .$$

For each of these values of v, the exponent $1/v$ is an integer. Using the binomial theorem, we have, for any such value of v,

$$(1 + v)^{\frac{1}{v}} = 1 + \frac{1}{v}v + \frac{\frac{1}{v}\left(\frac{1}{v} - 1\right)}{2!}v^2 + \frac{\frac{1}{v}\left(\frac{1}{v} - 1\right)\left(\frac{1}{v} - 2\right)}{3!}v^3 + \cdots + v^{\frac{1}{v}}.$$

$$= 1 + 1 + \frac{1 - v}{2!} + \frac{(1 - v)(1 - 2v)}{3!} + \cdots + v^{\frac{1}{v}}.$$

If v is allowed to approach zero through the sequence of values indicated above, this expansion is valid for each value of v; the number of terms in it increases indefinitely as $v \to 0$. This suggests that the required limit *may* be approximated to any desired degree of accuracy by setting $v = 0$ in the expansion and taking a sufficiently large number of terms of the resulting *infinite series.* This conclusion is valid although no rigorous justification of it can be given at this point. Denoting the value of the limit by e, we have

$$\lim_{v \to 0} (1 + v)^{\frac{1}{v}} = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots$$
$$= 2.718+.$$

The corresponding functions $y = e^x$ and $y = \log_e x$ are of course special cases of the functions $y = a^x$ and $y = \log_a x$

respectively. It will be found that these functions play a particularly important role in many applications of the calculus.

Logarithms to the base e are called *natural logarithms.* For reasons which will appear later, they will be used almost exclusively in our work. In this text the symbol "$\log x$," with no base indicated, will always mean the natural logarithm of x. Some writers prefer to use "$\ln x$" for this purpose.

PROBLEMS

1. (*a*) Prove that $a^m \cdot a^n = a^{m+n}$ if m and n are positive integers.
(*b*) Prove that $(a^m)^n = a^{mn}$ if m and n are positive integers.
(*c*) Prove that $(a \cdot b)^n = a^n b^n$ if n is a positive integer.

2. Simplify: $\sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}$.

3. (*a*) Prove that $\log_a (mn) = \log_a m + \log_a n$.
(*b*) Prove that $\log_a \frac{m}{n} = \log_a m - \log_a n$.
(*c*) Prove that $\log_a m^p = p \log_a m$.

4. Prove the formula for change of base, namely,

$$\log_b N = \frac{\log_a N}{\log_a b}.$$

Use this formula to compute $\log_{14} 67$.

5. Compute $\log_e 85$ using a table of common logarithms (to base 10). Check by referring to a table of natural logarithms.

6. Show that $\log_b a = \frac{1}{\log_a b}$.

7. Make sketches to show the general appearance of the curves $y = a^x$ and $y = a^{-x}$ if $a > 1$. What is the situation if $0 < a < 1$? If $a = 1$?

8. Sketch on the same axes the graphs of $y = 2^x$, $y = e^x$, and $y = 10^x$. How do they compare?

9. Sketch on the same axes the graphs of $y = \log_{10} x$ and $y = \log_e x$. How do they compare?

10. If one should draw the graph of $y = \log x$ and then double the ordinates, would the resulting graph be that of $y = \log x^2$? Explain.

11. Compute the value of e to 5 decimal places from the series.

12. Write out a definition of the logarithm of N to the base e in words. Is it obvious from the definition that $e^{\log N} = N$?

13. Evaluate $e^{\frac{1}{2}\log 64}$ without using tables. See Prob. 12.
14. Explain why $e^{2\log x} = x^2$. See Prob. 12.
15. Explain why $e^{-\frac{1}{2}\log x} = \frac{1}{\sqrt{x}}$. See Prob. 12.

46. The derivative of $\log_a v$.—To find the derivative of the function

$$y = \log_a x$$

we apply the fundamental differentiation process. Starting at any point P (Fig. 51) and letting x increase by a small amount Δx we have

$$\Delta y = \log_a (x + \Delta x) - \log_a (x)$$
$$= \log_a \frac{x + \Delta x}{x}$$
$$= \log_a \left(1 + \frac{\Delta x}{x}\right).$$
$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right).$$

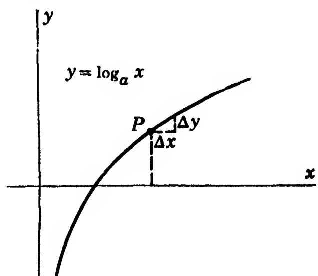

FIG. 51.

We cannot easily see what happens to the value of this fraction when $\Delta x \to 0$. However, multiplying numerator and denominator by x, it may be written in the form

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \frac{x}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right)$$
$$= \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

If now $\Delta x \to 0$, the quantity $\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$ approaches as a limit the number e because it is of the form $(1 + v)^{\frac{1}{v}}$ with v approaching zero. We have then

$$\frac{dy}{dx} = \frac{1}{x} \lim_{\Delta x \to 0} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$
$$= \frac{1}{x} \log_a e.$$

Finally, if we consider the function $y = \log_a v$ where v is a differentiable function of x, its derivative *with respect to v* is $\frac{1}{v}\log_a e$ by virtue of the above result. In order to obtain its derivative with respect to x we must multiply this result by dv/dx. That is,

(XIX) $$\frac{d}{dx}\log_a v = \frac{1}{v}\log_a e\,\frac{dv}{dx}.$$

For the special case of *natural* logarithms this becomes

(XIXs) $$\frac{d}{dx}\log v = \frac{1}{v}\frac{dv}{dx}.$$

Example 1

$$\begin{aligned} y &= \log(x^3 + 4). \\ \frac{dy}{dx} &= \frac{1}{x^3+4}\frac{d}{dx}(x^3+4) \qquad \text{[by (XIXs)]} \\ &= \frac{3x^2}{x^3+4}. \end{aligned}$$

Example 2

$$\begin{aligned} y &= \log_{10}\sin x. \\ \frac{dy}{dx} &= \frac{1}{\sin x}\log_{10} e\,\frac{d}{dx}(\sin x) \qquad \text{[by (XIX)]} \\ &= (\log_{10} e)\cot x. \end{aligned}$$

47. The derivative of a^v.—The derivative of this function is obtained easily from that of $\log_a v$ as follows: If

$$y = a^v$$

then

$$v = \log_a y,$$

and

$$\frac{dv}{dy} = \frac{1}{y}\log_a e \qquad \text{[by (XIX)].}$$

Then, using formula (VIII),

$$\begin{aligned} \frac{dy}{dv} &= y\frac{1}{\log_a e} \\ &= y\log_e a \\ &= a^v\log_e a. \end{aligned}$$

Finally, if v is a differentiable function of x we have the formula,

(XX) $$\frac{d}{dx}a^v = a^v \log a \frac{dv}{dx}.$$

If the constant is the particular number e, the formula becomes

(XXs) $$\frac{d}{dx}e^v = e^v \frac{dv}{dx}.$$

Example 1

$$y = e^{-x^2}.$$
$$\frac{dy}{dx} = e^{-x^2}\frac{d}{dx}(-x^2) \qquad \text{[by (XXs)]}$$
$$= -2xe^{-x^2}.$$

Example 2

$$y = 4^{3x-5}.$$
$$\frac{dy}{dx} = 4^{3x-5} \log 4\frac{d}{dx}(3x - 5) \qquad \text{[by (XX)]}$$
$$= 4^{3x-5}(3 \log 4).$$

Example 3

$$y = e^{-x} \log x.$$
$$\frac{dy}{dx} = e^{-x}\frac{d}{dx} \log x + \log x\frac{d}{dx}e^{-x}$$
$$= e^{-x}\frac{1}{x} + (\log x)e^{-x}(-1)$$
$$= e^{-x}\left(\frac{1}{x} - \log x\right).$$

PROBLEMS

1. Show that the slope of the curve $y = \log_a x$ at any point is inversely proportional to the abscissa of the point.

2. Show that the natural logarithm of x increases only $\frac{1}{100}$ as fast as x when $x = 100$.

3. A quantity Q varies with the time according to the law $Q = Ae^{kt}$ where A and k are constants. Show that the rate at which Q is changing at any instant is proportional to the value of Q at that instant.

4. If x is increasing at 4 units per minute at what rate is the value of the function 2^{-x^2} changing when $x = 2$?

Differentiate the following functions:

5. $y = \log x^3$. **6.** $y = \log \sqrt{3x + 4}$.

7. $y = \log \dfrac{x^2 + 4}{x^2 - 4}$. **8.** $y = \log \sin x$.

9. $y = \log_{10}{}^2 x$. **10.** $y = \log_a \sqrt{r^2 - x^2}$.

11. $y = \log \sec^2 x$. **12.** $y = \log (x + \sqrt{x^2 - a^2})$.

13. $y = \log (\sec x + \tan x)$. **14.** $y = \log \log x$.

In each of the following find dy/dx using the method suggested by the hint in Prob. 15:

15. $y = \log (x\sqrt{x^2 - 4})$. HINT: First write the function in the form $y = \log x + \frac{1}{2} \log (x^2 - 4)$.

16. $y = \log \sqrt{\dfrac{1 + t}{1 - t}}$. **17.** $y = \log \sqrt{\dfrac{1 + x^2}{1 - x^2}}$.

18. $y = \log (x^2 \sin x)$. **19.** $y = \log [(x^2 - 1)\sqrt{x + 1}]$.

20. Show that if $y = \frac{1}{2}[\sec \theta \tan \theta + \log (\sec \theta + \tan \theta)]$, then $dy/d\theta = \sec^3 \theta$.

Differentiate the following functions:

21. $y = 4e^{\frac{1}{2}x}$. **22.** $y = e^{-x^2}$.

23. $y = 2^{3x}$. **24.** $y = 10^{x^2-1}$.

25. $y = \dfrac{3^{x^2}}{9}$. **26.** $y = \dfrac{1}{(2e)^x}$.

27. $y = xe^{-x}$. **28.** $y = 4x^2e^{x^2}$.

29. $y = e^x \log x$. **30.** $y = 2e^{-x} \sin x$.

31. $y = \frac{1}{2}(e^x + e^{-x})$. **32.** $y = \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

33. $y = e^{-\frac{1}{2}x} \cos \dfrac{\pi x}{2}$.

For each of the following curves determine the maximum, minimum, and inflection points. Sketch the curves:

34. $y = e^{-x} \sin x$ $\qquad 0 \leqq x \leqq 2\pi$.

35. $y = e^{-\pi x} \sin \pi x$ $\qquad 0 \leqq x \leqq 2$.

36. $y = 4e^{-\frac{x}{2}} \cos \dfrac{\pi x}{2}$ $\qquad 0 \leqq x \leqq 4$.

37. $y = \dfrac{x}{\log x}$. **38.** $y = x \log x$.

39. $y = xe^{-x}$. **40.** $y = e^{-x^2}$.

41. At what angle do the curves $y = \log x$ and $y = \log x^2$ intersect?

42. Show that the catenary $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ is everywhere concave upward. Find its minimum point; sketch the curve.

48. Logarithmic differentiation.—If a function which is to be differentiated is a product of several factors, the work may be materially simplified by taking the natural logarithm of the function *before* differentiating.

Example

Differentiate $y = \dfrac{x\sqrt{4x+3}}{(3x+1)^2}$.

Solution

Taking the natural logarithm of each side of the given equation we may write

$$\log y = \log x + \tfrac{1}{2}\log(4x+3) - 2\log(3x+1).$$

Then using implicit differentiation we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{2}{4x+3} - \frac{6}{3x+1};$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{2}{4x+3} - \frac{6}{3x+1}\right).$$

49. The derivative of u^v.—The derivative of a function of the form

$$y = u^v,$$

where u and v are both differentiable functions of x, can be found by the method of logarithmic differentiation.

Example

Differentiate $y = (x+1)^x$.

Solution

Taking the natural logarithm of each side we have

$$\log y = x\log(x+1).$$

Differentiating this relation implicitly with respect to x, we find

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x+1} + \log(x+1);$$

$$\frac{dy}{dx} = y\left[\frac{x}{x+1} + \log(x+1)\right].$$

The student should notice that neither the formula for the derivative of v^n nor that for the derivative of a^v applies to functions of this type. Why?

50. The derivative of v^n for any value of n.—In Chap. IV the proof of the formula for the derivative of v^n was given only for n a positive integer. The proof for *any* value of n may now be given as follows. Let

$$\begin{aligned} y &= v^n; \\ \log y &= n \log v \\ \frac{1}{y}\frac{dy}{dx} &= n\frac{1}{v}\frac{dv}{dx} \\ \frac{dy}{dx} &= \frac{ny}{v}\frac{dv}{dx}. \end{aligned}$$

But, since

$$y = v^n,$$

this reduces to

$$\frac{dy}{dx} = nv^{n-1}\frac{dv}{dx}.$$

PROBLEMS

Find the derivative of each of the following functions using logarithmic differentiation:

1. $y = x\sqrt{x^2 - 1}$.
2. $y = t\sqrt{t + 1}\sqrt[3]{t - 1}$.
3. $y = \sqrt{\dfrac{(x + 2)(x + 3)}{x(x + 1)}}$.
4. $y = \sqrt{\dfrac{1 + x^2}{1 - x^2}}$.
5. $y = \dfrac{(a^2 + x^2)^{\frac{1}{2}}}{x(a^2 - x^2)^{\frac{1}{2}}}$.
6. $y = \dfrac{3x^2 + 4}{\sqrt[3]{x^2 - 6}}$.
7. $y = x^x$.
8. $y = x^{\frac{1}{x}}$.
9. $y = x^{\log x}$.
10. $y = x^{\sin x}$.
11. $y = (\log x)^x$.
12. $y = \left(\dfrac{2}{x}\right)^x$.

13. Using logarithmic differentiation show that the derivative of u^v is given by the formula

$$\frac{d}{dx}u^v = vu^{v-1}\frac{du}{dx} + u^v \log u\frac{dv}{dx}.$$

Note that this amounts to differentiating the function as if v were constant and then as if u were constant and adding the results.

14. Find the derivative of x^x both with and without using the formula of Prob. 13 and show that the results are the same.

15. Find the derivative of $(\log x)^x$ both with and without using the formula of Prob. 13 and show that the results are the same.

CHAPTER IX

DERIVATIVE OF ARC. CURVATURE

51. Smooth curves.—Suppose that a given curve has, at every point of a certain interval, a tangent line; suppose furthermore that the tangent line turns continuously as it traverses the curve. The curve is then said to be *smooth* in the interval. Throughout this chapter it will be assumed that the curves under discussion have this property of smoothness.

52. Derivative of arc.—Think of a point P moving along the smooth curve whose equation is $y = f(x)$. Denote the

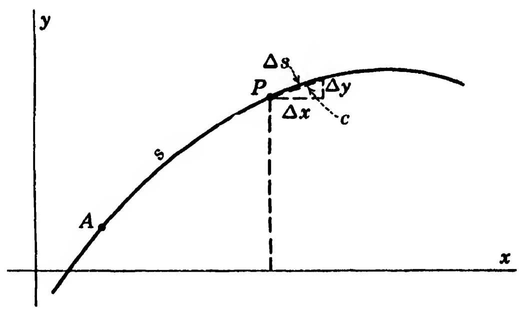

FIG. 52.

length of arc, measured from an arbitrarily chosen fixed point A, to P by s. We wish to determine the rate at which this length s is changing with respect to x; *i.e.*, we wish to find the rate, measured with respect to x, at which the arc itself is being traced when the moving point is at any position $P(x, y)$ on the curve.

We may denote by Δs the distance, measured along the arc, which the point P must traverse in order that its abscissa may increase by a small amount Δx. The required rate is then given by

$$\lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \frac{ds}{dx}.$$

In order to compute this limit we observe that (Fig. 52),

$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{c} \frac{c}{\Delta x}.$$

From the figure it is evident that $c^2 = \overline{\Delta x}^2 + \overline{\Delta y}^2$; hence,

$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{c} \frac{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}}{\Delta x}$$

$$= \frac{\Delta s}{c}\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

It is fairly obvious that as $\Delta x \to 0$, the quotient $\Delta s/c$ approaches 1 as a limit. This can be proved but we shall not give the proof here. We have then

$$\lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta s}{c} \cdot \lim_{\Delta x \to 0} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

or

$$\text{(XXI)} \qquad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

This expression gives the rate at which the *arc* is lengthening with respect to x in the same way that the value of dy/dx gives the rate at which the *ordinate* is increasing or decreasing relative to x. It is of course measured in units such as inches per inch; *i.e.*, in inches moved along the arc per inch moved in the direction of the x-axis.

The student may easily show that a corresponding expression for the derivative *with respect to* y of the length of arc is

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

Example

Find ds/dx for the circle $x^2 + y^2 = 25$. If a point moves along the arc from $(0, 5)$ to $(5, 0)$ in such a way that its abscissa increases uni-

formly at 2 units per minute, at what rate is the arc being traced when it passes through (3, 4)?

Solution

$$x^2 + y^2 = 25.$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$$\frac{ds}{dx} = \sqrt{1 + \frac{x^2}{y^2}} = \frac{\sqrt{y^2 + x^2}}{y} = \frac{5}{y}.$$

The value of ds/dx at the point (3, 4) is

$$\left.\frac{ds}{dx}\right]_{3,4} = \frac{5}{4}.$$

This means that the arc is being traced $\frac{5}{4}$ times as fast as the abscissa is changing, or at a rate of $2\frac{1}{2}$ units per minute. This is of course the speed of the moving point at this instant.

PROBLEMS

1. Show that $ds/dy = \sqrt{1 + (dx/dy)^2}$.

2. Show that when a point, in moving along a curve, passes through a maximum or minimum point where $dy/dx = 0$, the arc is increasing at the same rate as x. Does this fact appear obvious from the graph?

3. In deriving the formula for ds/dx, only the positive sign was used before the radical. Under what condition should the negative sign be used? HINT: If A (Fig. 52) is to the right of P then s decreases as x increases.

Find ds/dx for each of the following curves:

4. $y = \sin x$.

5. $y = \sqrt{x}$.

6. $y = \log x$.

7. $x^2 + y^2 = r^2$.

8. $y = \text{arc} \cos x$.

9. $y = \log \cos x$.

10. Compute the value of $\left.\frac{ds}{dx}\right]_{x=5}$ for the curve $y = x^{3/2}$. Explain the result, stating the units in which it is expressed.

11. A point moves along the curve $y = \text{arc} \sin x$. What is its speed when it passes through the origin if its abscissa is increasing at 3 units per second?

12. For what points on a curve is the value of ds/dx equal to that of ds/dy?

13. Show that if θ is the inclination of the tangent line to a curve at P, then the value of $dx/ds = \cos \theta$ and $dy/ds = \sin \theta$ while $dy/dx = \tan \theta$.

14. Show that for the catenary $y = \frac{e^x + e^{-x}}{2}$, the value of ds/dx at any point is equal to y.

53. Curvature of a circle.—In Fig. 53 the tangent line to the circle at P makes an angle θ with the x-axis. In moving to the position P' it turns through an angle $\Delta\theta$. The amount of turning *per unit distance moved along the arc* is obviously

$$\frac{\Delta\theta}{\Delta s} = \frac{\Delta\theta}{R\,\Delta\theta} = \frac{1}{R}.$$

This is called the *curvature* of the circular arc. Thus the curvature of a circle whose radius is 4 ft. is

$$K = \frac{1}{R} = \frac{1}{4} \text{ radian per foot.}$$

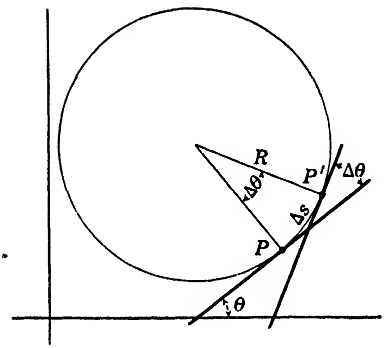

FIG. 53.

This means of course that the tangent line, in traversing the circle, turns at a rate of $\frac{1}{4}$ radian per foot moved along the arc.

54. Curvature of any curve.—Consider now any smooth curve. A tangent line traversing it turns at a *variable* rate, with respect to the distance moved along the arc, instead of at a constant rate as it obviously does in the case of the circle. The value of this rate at any point P on the curve is called the *curvature* of the arc at P.

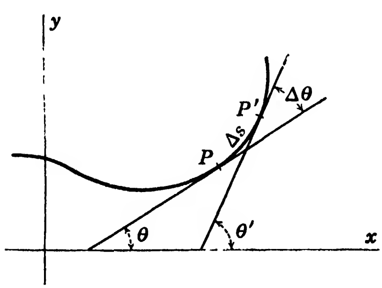

FIG. 54.

In order to compute this rate we first observe from Fig. 54 that the average curvature of the arc from P to a neighboring point P' is expressed by the fraction

$$\frac{\Delta\theta}{\Delta s}$$

where $\Delta\theta = \theta' - \theta$ is the angle turned through by the tangent line moving from P to P', and Δs is the corresponding length of arc. It is natural that we should define the curvature at P as the *limit* approached by this fraction as $\Delta s \to 0$; *i.e.*,

$$K = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}.$$

To obtain an expression for the curvature in terms of dy/dx and d^2y/dx^2 we need only to notice from the figure that the value of dy/dx at P is equal to $\tan\theta$; hence,

$$\theta = \arc\tan\frac{dy}{dx}.$$

Differentiating carefully *with respect to s* we find

$$\begin{aligned}\frac{d\theta}{ds} &= \frac{d\theta}{dx}\cdot\frac{dx}{ds}\\ &= \frac{d}{dx}\left(\text{arc tan}\,\frac{dy}{dx}\right)\cdot\frac{dx}{ds}\\ &= \frac{\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)}{1+\left(\dfrac{dy}{dx}\right)^2}\cdot\frac{dx}{ds}\\ &= \frac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}\cdot\frac{dx}{ds}.\end{aligned}$$

Since

$$\frac{dx}{ds} = \frac{1}{\dfrac{ds}{dx}} = \frac{1}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}},$$

this reduces to

$$\text{(XXII)} \qquad \frac{d\theta}{ds} = K = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

Example

Find the curvature of the cubical parabola $y = x^3$ at (1, 1).

Solution

$$\frac{dy}{dx} = 3x^2 \qquad \left.\frac{dy}{dx}\right]_{1,1} = 3.$$

$$\frac{d^2y}{dx^2} = 6x \qquad \left.\frac{d^2y}{dx^2}\right]_{1,1} = 6.$$

$$K = \frac{6}{(1 + 3^2)^{\frac{3}{2}}} = \frac{6\sqrt{10}}{100} = 0.1897.$$

It should be observed that the sign of K, as determined by (XXII), is always the same as that of d^2y/dx^2; *i.e.*, the sign of K will be *plus* at any point where the curve is concave upward and *minus* at any point where it is concave downward.

55. Transition curves.—The curvature of a curve at a point P, being the rate at which the tangent line is turning

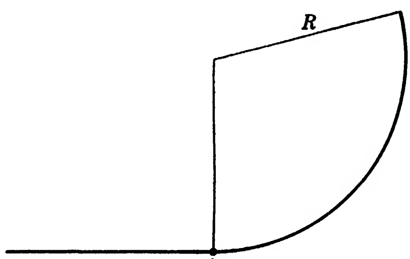

FIG. 55.

with respect to the distance moved along the arc, obviously measures the rate at which a particle traversing the curve is changing its direction. Suppose now that in laying out a railroad curve one joined the straight track directly to a circular arc of radius R (in feet) as indicated in Fig. 55. The curvature would of course change suddenly from 0 to $1/R$ radians per foot at the junction A. When a train was passing this point there would be brought into play certain suddenly applied radial forces which would result in undesirable stresses as well as disagreeable lurching, etc.

It is much more desirable to have the straight track joined to the circular arc by a section AB (Fig. 56) whose curvature increases gradually from 0 at A to $1/R$ at B. Such a curve is called a *transition* or easement curve. The cubical parabola has often been used as a transition curve although it appears to have no property which makes it particularly desirable for this purpose other than the fact that its equation is simple. Probably the best curve for the purpose is one whose curvature increases linearly with the distance s measured along the curve from A to B. Such a curve is called a transition spiral. Its equation

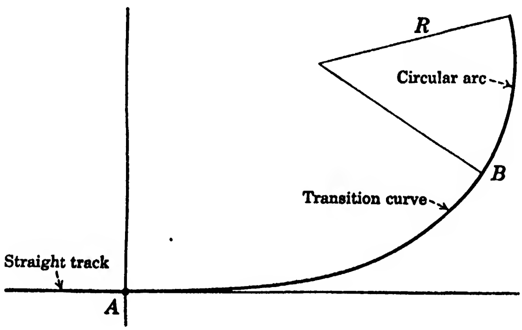

FIG. 56.

cannot be expressed in simple form but methods of laying out the curve, with sufficient accuracy for practical purposes, are given in books on railway surveying.*

56. Radius of curvature.—We have seen that the radius of a circle is equal to the reciprocal of its curvature. By analogy we may define the *radius of curvature* of any curve at a point P as the reciprocal of its curvature at this point. This radius is of course given by

$$\text{(XXIII)} \qquad R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

* See, for example, A. N. Talbot, "The Railway Transition Spiral," McGraw-Hill Book Company, Inc., 1927.

By measuring off along the normal to the curve at P, on the concave side, a distance R as determined by (XXIII), one can locate a point C called the *center of curvature.* The circle with center at C and radius R is called the *circle of curvature.* Of the indefinitely large number of circles that can be drawn tangent to the curve at P, this is the only one whose curvature is the same as that of the curve at the point of contact. It can be shown that this circle "fits" the curve more closely in the neighborhood of P than any other circle—just as the tangent line fits it more closely than any other line.

Another definition of the circle of curvature at P is as follows: Suppose that we pass a circle through P and two arbitrarily selected neighboring points P' and P'' on the curve. The limiting position of this circle as P' and P'' both approach P along the curve can be shown to be identical with that of the circle of curvature as defined above. This definition of the circle suggests an approximate method of constructing it graphically.

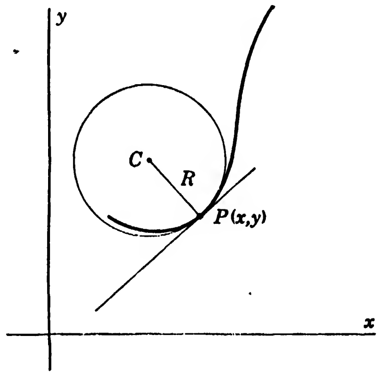

FIG. 57.

Another formula for R, which is equivalent to (XXIII), and which may be used in case it is more convenient to take derivatives with respect to y, is

$$\text{(XXIV)} \qquad R = -\frac{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}.$$

This formula should also be used in finding R at a point where the tangent line to the curve is parallel to the y-axis; for at such a point $dx/dy = 0$.

Example

Compute R at $(a, 0)$ on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Solution

Differentiating implicitly *with respect to y*, we have

$$2b^2x\frac{dx}{dy} + 2a^2y = 0$$

$$\frac{dx}{dy} = -\frac{a^2y}{b^2x}; \qquad \left.\frac{dx}{dy}\right]_{a,0} = 0.$$

$$\frac{d^2x}{dy^2} = \left(-\frac{a^2}{b^2}\right)\left(\frac{x - y\frac{dx}{dy}}{x^2}\right); \qquad \left.\frac{d^2x}{dy^2}\right]_{a,0} = -\frac{a}{b^2}.$$

$$R = -\frac{(1+0)^{\frac{3}{2}}}{-\frac{a}{b^2}} = \frac{b^2}{a}.$$

PROBLEMS

1. What is the curvature in radians per foot and in degrees per foot of a circular arc whose radius is 100 ft.?

2. At what rate, in degrees per 100 ft. of arc, is a train changing its direction when it is running on a circular curve of radius 500 ft.? NOTE: This rate is called by surveyors the *degree-of-curve*. Thus a 5° curve is one whose curvature is 5° per 100 feet.

Compute K and R for each of the following curves at the point indicated:

3. $y = x^2$; $(1, 1)$.

4. $y = x^2 - 2x - 3$; $(1, -4)$.

5. $y = x^3 - 5x^2 + 12$; $(3, -6)$.

6. $x^2 = 8 - 4y$; $(2, 1)$.

7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $(0, b)$ and $(a, 0)$.

8. $4x^2 - y^2 = 20$; $(3, 4)$.

9. $y = \log x$; $(e, 1)$.

10. $y = \sin x$; $\left(\frac{\pi}{2}, 1\right)$.

11. $y = \cos x$; $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.

12. $y = \log \cos x$; (x_1, y_1).

13. $y = \log \sec x$; (x_1, y_1).

14. $y = e^{-x^2}$; $(0, 1)$.

15. Sketch the curve $y = \frac{1}{3}x^3$. Obtain an expression for its curvature at any point. Discuss the way in which its curvature varies.

16. Express the radius of curvature of $y = \log x$ as a function of x. For what value of x is $|R|$ a minimum and what is this smallest value?

17. Compute the least radius of curvature of the parabola

$$y = x^2 - 2x - 3.$$

18. Compute the radius of curvature of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ at the minimum point on the curve.

19. Find the curvature of $y = 12x - x^3$ at the maximum point on the curve.

20. Show that the curvature of $y = f(x)$ at $P(x, y)$ is given by $K = \frac{d^2y}{dx^2} \cos^3 \theta$ where θ is the inclination of the tangent line at P.

21. Derive expressions for the coordinates α and β of the center of curvature of $y = f(x)$ at $P(x, y)$ in terms of x, y, dy/dx, and d^2y/dx^2.

22. Find the coordinates of the center of curvature of the curve $y = \log x$ at $(1, 0)$.

23. Derive formula (XXIV) from (XXIII) by substituting for dy/dx and d^2y/dx^2 their values in terms of dx/dy and d^2x/dy^2. HINT: We know that $\frac{dy}{dx} = \frac{1}{dx/dy}$. Differentiate this very carefully with respect to x to obtain d^2y/dx^2 in terms of dx/dy and d^2x/dy^2. Note that $\frac{d^2y}{dx^2}$ is *not* equal to $\frac{1}{d^2x/dy^2}$.

CHAPTER X

PARAMETRIC EQUATIONS

57. Introduction.—In many cases it is convenient to define y indirectly as a function of x by means of two equations of the form

$$y = G(\theta) \qquad x = H(\theta)$$

where θ is a third variable called a *parameter*. The direct relation, $y = f(x)$, would result from the operation of

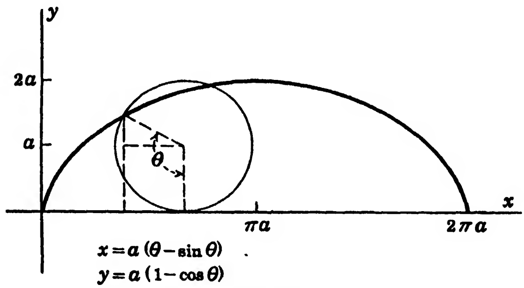

FIG. 58.

eliminating θ between these equations. This relation, if obtained, might be very complicated, and often it is simpler to deal with the relation in the original parametric form.

The problems of determining maximum and minimum points, inflection points, curvature, etc., for a curve defined in this way, present no new principles. One proceeds in the usual way after having found the necessary derivatives. We need then only show how to obtain the successive derivatives *without eliminating the parameter*.

58. To find dy/dx and d^2y/dx^2.—In order to illustrate the procedure let us compute dy/dx and d^2y/dx^2 for the cycloid (Fig. 58) whose parametric equations are

$$y = a(1 - \cos\theta), \qquad x = a(\theta - \sin\theta).$$

Regarding θ as the independent variable, we have

$$\frac{dy}{d\theta} = a \sin \theta, \qquad \frac{dx}{d\theta} = a(1 - \cos \theta).$$

Then since

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}$$

we obtain

$$\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta}.$$

It is evident that this procedure always gives dy/dx in terms of the parameter θ rather than in terms of x.

To obtain d^2y/dx^2 we must proceed very carefully, remembering that d^2y/dx^2 is the derivative *with respect to* x of dy/dx. That is,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{\sin \theta}{1 - \cos \theta}\right). \end{aligned}$$

In order to obtain the derivative of $\dfrac{\sin \theta}{1 - \cos \theta}$ *with respect to* x we may take its derivative *with respect to* θ and multiply by $d\theta/dx$. Thus,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{\sin \theta}{1 - \cos \theta}\right) \cdot \frac{d\theta}{dx} \\ &= \left(-\frac{1}{1 - \cos \theta}\right) \cdot \frac{d\theta}{dx}; \end{aligned}$$

since

$$\frac{d\theta}{dx} = \frac{1}{\dfrac{dx}{d\theta}} = \frac{1}{a(1 - \cos \theta)},$$

this reduces to

$$\frac{d^2y}{dx^2} = -\frac{1}{a(1 - \cos\theta)^2}.$$

PROBLEMS

Find dy/dx and d^2y/dx^2 for each of the following curves:

1. $y = a \sin\theta$, $x = a \cos\theta$.
2. $y = 3 \sin\theta$, $x = 4 \cos\theta$.
3. $y = 4 \sin^2 t$, $x = 8 \cos t$.
4. $y = 2 \sin 2\theta$, $x = 4 \cos\theta$.
5. $y = \sin 2\theta$, $x = \tan\theta$.
6. $y = 4a \sin^3\theta$, $x = 4a \cos^3\theta$.

7. Find the equations of tangent and normal lines to the curve

$$x = t^2 - 1, \qquad y = t^4 - 4,$$

at the point where $t = 2$. Eliminate the parameter and sketch the curve.

8. Locate the minimum point on the curve of Prob. 7, and compute the radius of curvature at this point.

9. Show that $d^2y/dx^2 = 0$ at every point of the curve $x = 3 \sin^2\theta$, $y = 6 \cos^2\theta$. Sketch the graph.

10. Show that at any point on the curve $x = \log t$, $y = t + 4$, the values of dy/dx and d^2y/dx^2 are equal. Eliminate the parameter and sketch the curve.

11. Find the radius of curvature at any point of the curve $y = 2p \tan\theta$, $x = p \tan^2\theta$. Show by eliminating θ that the curve is a parabola.

12. For the curve $x = a \sec\theta$, $y = b \tan\theta$, compute d^2y/dx^2. Then eliminate the parameter and compute d^2y/dx^2 from the resulting equation. Show that the results are equivalent. Sketch the curve.

13. Locate the maximum, minimum, and inflection points on the curve $x = 4 \cot\theta$, $y = 4 \sin^2\theta$. Sketch the curve.

14. Compute the radius of curvature at a maximum point on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

15. Let y be defined as a function of x by the equations $x = f(t)$, $y = g(t)$. Show that

$$\frac{d^2y}{dx^2} = \frac{\begin{vmatrix} \dfrac{dx}{dt} & \dfrac{dy}{dt} \\[2mm] \dfrac{d^2x}{dt^2} & \dfrac{d^2y}{dt^2} \end{vmatrix}}{\left(\dfrac{dx}{dt}\right)^3}.$$

59. Velocity and acceleration in curvilinear motion.— Suppose that a point moves in the xy-plane in such a way

that its coordinates at any time t are given by

$$x = G(t), \qquad y = H(t).$$

It will describe a curve whose equation, in the form $y = f(x)$, would be obtained by eliminating t.

The velocity of the moving point at any instant may be determined by finding its components parallel to the coordinate axes. For this purpose we observe that the value of dx/dt at any instant is the time rate at which the x-coordinate is changing; *i.e.*, it is the rate at which the projection of the point on the x-axis is moving along this axis. It is therefore the component of the velocity in this

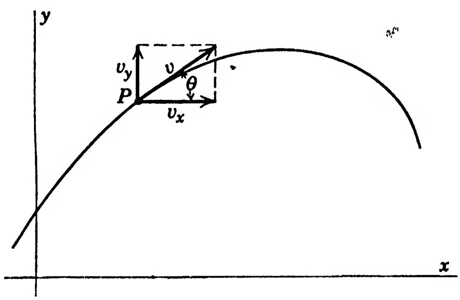

FIG. 59.

direction. Similarly, the value of dy/dt is the component of the velocity in the y-direction. Denoting these components by v_x and v_y we have

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}.$$

The magnitude and direction of the velocity are then given by

$$v = \sqrt{v_x^2 + v_y^2}, \qquad \theta = \text{arc tan}\,\frac{v_y}{v_x}.$$

It should be observed that, since

$$\frac{v_y}{v_x} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{dy}{dx},$$

the direction of the velocity vector is that of the tangent to the path (see Fig. 59).

In a similar manner the *acceleration* of the moving point at any instant may be determined by finding its components. Thus, the time rate at which v_x is changing is the x-component of the acceleration while the rate at which

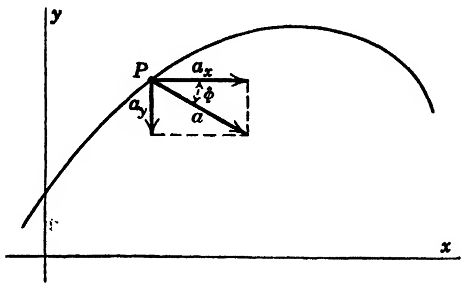

FIG. 60.

v_y is changing is the y-component. Denoting these components by a_x and a_y we have

$$a_x = \frac{d^2x}{dt^2}, \qquad a_y = \frac{d^2y}{dt^2}.$$

The magnitude and direction of the acceleration are then expressed by

$$a = \sqrt{a_x^2 + a_y^2}, \qquad \phi = \text{arc tan}\,\frac{a_y}{a_x}.$$

It should be carefully observed that, since

$$\frac{a_y}{a_x} = \frac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}} \neq \frac{dy}{dx},$$

the direction of the acceleration vector (Fig. 60) is *not* that of the tangent to the path. It can be shown that it is always directed toward the concave side of the path and, in particular, is *normal* to the path if the *speed* of the moving point is *constant.*

Example

A point moves in the xy-plane so that at any time t its coordinates are

$$x = t^2 - 2, \qquad y = 3t^2 - \frac{t^4}{4}.$$

Compute the magnitude and direction of its velocity and acceleration when $t = 2$. Find the Cartesian equation of its path.

Solution

$$v_x = \frac{dx}{dt} = 2t; \qquad v_y = \frac{dy}{dt} = 6t - t^3;$$
$$v_x]_{t=2} = 4. \qquad v_y]_{t=2} = 4.$$
$$v = \sqrt{4^2 + 4^2} = 4\sqrt{2},$$
$$\theta = \text{arc tan } 1.$$
$$a_x = \frac{d^2x}{dt^2} = 2; \qquad a_y = \frac{d^2y}{dt^2} = 6 - 3t^2;$$
$$a_x]_{t=2} = 2. \quad a_y]_{t=2} = -6.$$
$$a = \sqrt{2^2 + (-6)^2} = 2\sqrt{10},$$
$$\varphi = \text{arc tan } (-3).$$

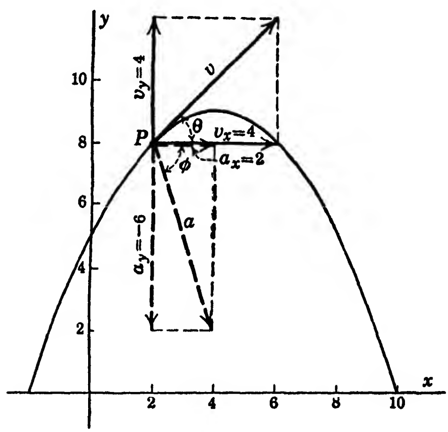

FIG. 61.

Eliminating t we find that the path described by the moving point is the parabola

$$4y = 20 + 8x - x^2.$$

The position of the point on the path when $t = 2$, and the corresponding vectors representing v_x, v_y, v, and a_x, a_y, a, are shown in Fig. 61.

PROBLEMS

1. The velocity of a point which is moving along a straight or curved path may be defined as the *vector* whose magnitude is ds/dt and whose direction is that of the tangent to the path. Show that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

HINT: $\dfrac{ds}{dt} = \dfrac{ds}{dx} \cdot \dfrac{dx}{dt}$ and $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.

2. In the above illustrative example find v and a when $t = 3$. Sketch the curve and draw the vectors.

In each of the following, compute the values of v and a at the instant indicated. Find the Cartesian equation of the path, sketch it, and draw the vectors.

3. $x = 2t^2 - 4$, $y = t^4 - 8t^2$; $\quad t = 2$.

4. $x = 2t + 1$, $y = t^3$; $\quad t = 1$.

5. $x = 4 \cos t$, $y = 4 \sin t$; $\quad t = \dfrac{\pi}{3}$.

6. A point moves so that its coordinates at any time t are $x = 2 \sin t$, $y = 2(1 - \cos t)$. Show that its velocity and acceleration are constant in magnitude. Find the Cartesian equation of the path and describe the motion.

7. A point moves so that its coordinates at any time t are given by $x = 8 - 8 \sin t$, $y = 4 \cos t$. Find its velocity and acceleration when $t = \pi/2$. Find the Cartesian equation of the curve, sketch it, and draw the v and a vectors.

8. A point starts at $(r, 0)$ when $t = 0$ and moves around the circle $x^2 + y^2 = r^2$ at a constant angular velocity of ω radians per second. Show that its coordinates at any time t are, $x = r \cos \omega t$, $y = r \sin \omega t$. Show that its acceleration is constant in magnitude and is always directed toward the center of the circle.

9. Solve Prob. 8 for the case in which the radius drawn to the initial position of the moving point makes an angle α with the x-axis.

10. A wheel of radius r rolls along a straight line at a constant angular velocity of ω radians per second. Consider the point A which is in contact with the ground when $t = 0$ and show that its coordinates after t seconds are

$$x = r(\omega t - \sin \omega t), \qquad y = r(1 - \cos \omega t).$$

Show that its acceleration at any instant is $r\omega^2$ and is directed toward

the center of the wheel. The coordinate axes are assumed as shown in Fig. 58.

11. A bug starts at the center of a wheel of radius 4 ft. and crawls out along a spoke at 2 ft. per minute while the wheel rolls along a straight line at $\frac{1}{4}$ r.p.m. Derive parametric equations of the path followed by the bug in space. Find the components of his velocity at the end of 1 min.

12. If a projectile is fired from A (Fig. 62) with initial velocity v_0 at an angle α, its position at the end of t seconds (neglecting air resistance) is given by

$$x = (v_0 \cos \alpha)t$$
$$y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2.$$

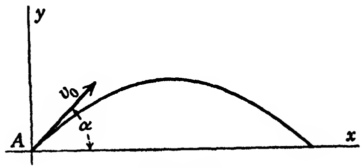

FIG. 62.

Find the components of its velocity and acceleration at any time. Show by eliminating t that the path is a parabola.

13. Using the data of Prob. 12, find the maximum height reached by the projectile. Find also the horizontal distance traveled.

14. Using the data of Prob. 12, find the angle α for which the horizontal distance traveled (range) is a maximum.

CHAPTER XI

COLLECTION OF FORMULAS
REVIEW PROBLEMS

60. Review.—The purpose of this chapter is to present an opportunity for a brief review. In this connection the student should again study the definition and the physical and geometrical interpretation of the derivative. He should make certain that he has a clear understanding of the fundamental process of differentiation, and its use in deriving the various formulas for differentiation. He should also make a step by step outline of the procedure to be followed in solving the important types of applied problems. The first 15 problems in the review set are designed to cover some of these points. The others may be used for review practice in formal differentiation and in solving the various types of applied problems. The formulas are listed below for easy reference.

FORMULAS

(I) $$\frac{dc}{dx} = 0.$$

(II) $$\frac{dx}{dx} = 1.$$

(III) $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

(IV) $$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

(V) $$\frac{d}{dx}(v^n) = nv^{n-1}\frac{dv}{dx}.$$

(VI) $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

(VII) $$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

(VIII) $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

(IX) $$\frac{d}{dx} \sin v = \cos v \frac{dv}{dx}.$$

(X) $$\frac{d}{dx} \cos v = -\sin v \frac{dv}{dx}.$$

(XI) $$\frac{d}{dx} \tan v = \sec^2 v \frac{dv}{dx}.$$

(XII) $$\frac{d}{dx} \cot v = -\csc^2 v \frac{dv}{dx}.$$

(XIII) $$\frac{d}{dx} \sec v = \sec v \tan v \frac{dv}{dx}.$$

(XIV) $$\frac{d}{dx} \csc v = -\csc v \cot v \frac{dv}{dx}.$$

(XV) $$\frac{d}{dx} \text{arc} \sin v = \frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}.$$

(XVI) $$\frac{d}{dx} \text{arc} \cos v = -\frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}.$$

(XVII) $$\frac{d}{dx} \text{arc} \tan v = \frac{\frac{dv}{dx}}{1 + v^2}.$$

(XVIII) $$\frac{d}{dx} \text{arc} \cot v = -\frac{\frac{dv}{dx}}{1 + v^2}.$$

(XIX) $$\frac{d}{dx} \log_a v = \frac{1}{v} \log_a e \frac{dv}{dx}.$$

(XIXs) $$\frac{d}{dx} \log v = \frac{1}{v}\frac{dv}{dx}.$$

(XX) $$\frac{d}{dx} a^v = a^v \log a \frac{dv}{dx}.$$

(XXs) $$\frac{d}{dx} e^v = e^v \frac{dv}{dx}.$$

(XXI) $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

(XXII) $$K = \frac{d\theta}{ds} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

(XXIII) $$R = \frac{1}{K} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

REVIEW PROBLEMS

1. Write out a definition of the derivative of a function in words and in symbols. Illustrate graphically.

2. Show why the value of dy/dx at a point P on the curve $y = f(x)$ represents the slope of the tangent line.

3. Using the fundamental differentiation process find the derivative of x^3; of $\cos x$.

4. Show that if a cube is expanding, its volume is increasing at a rate of 12 cu. in. per inch of increase in the edge at the instant when each edge is 2 in. long. If the edge is increasing at $\frac{1}{4}$ in. per minute, at what time rate is the volume increasing?

5. Show that $\tan x$ increases four times as rapidly as x when $x = \pi/3$. If x increases from 60° to 60° 10′ (10′ = 0.003 radian), what should be the *approximate* increase in the value of $\tan x$? Check your answer by referring to a table.

6. Show that the natural logarithm of x increases at a rate of $\frac{1}{100}$ unit per unit increase in x when $x = 100$.

7. How many of the formulas from (IX) to (XVIII) inclusive must be derived by the fundamental differentiation process? How are the others obtained?

8. Show that the derivative of $\sec x$ with respect to $\tan x$ is $\sin x$; hence, show that $\sec x$ increases half as rapidly as $\tan x$ when $x = \pi/6$. What is the situation when x is near $\pi/2$?

9. Derive the formula for the derivative of $\tan x$ both with and without using the fundamental differentiation process.

Outline the procedure to be followed in solving each of the following problems. Number the steps:

10. To find the equation of the line which is tangent to the curve $y = f(x)$ at a given point $P(x_1, y_1)$ on the curve.

11. To determine the angle at which two given curves, $y = f(x)$ and $y = \varphi(x)$, intersect.

12. To determine the maximum and minimum values of a given function.

13. To determine the inflection points on a given curve.

14. To compute the curvature of a given curve at a given point.

15. To determine the point on a given curve at which the radius of curvature is a maximum or minimum.

In each of the following, find dy/dx:

16. $y = 4x^3 - 7x + 2.$

17. $y = (3x^2 + 5)^2.$

18. $y = \dfrac{4 - 2x}{3 - x}.$

19. $y = x\sqrt{a^2 - x^2}.$

20. $y = \dfrac{3x^2}{2x + 1}.$

21. $y = \sqrt{\dfrac{3x + 1}{3x - 1}}.$

22. $y = \log (x^2\sqrt{2x^2 + 4x}).$

23. $y = x^2 \log x.$

24. $y = \log^2 (3x).$

25. $y = xe^{-x^2}.$

26. $y = 10^{x+4}.$

27. $y = (\log x)^x.$

28. $y = (x + 4)^x.$

29. $y = \sin^2 2x.$

30. $y = 3 \cos^2 \frac{1}{2}x.$

31. $y = \sin x \cos^2 x.$

32. $y = \frac{1}{2}[\sec x \tan x + \log (\sec x + \tan x)].$

33. $y = \arcsin \sqrt{x}.$

34. $y = \arctan \dfrac{20}{x}.$

35. $y = \arctan \dfrac{r - x}{r}.$

36. $9x^2 + 16y^2 = 144.$

37. $x^2y = 1 - y.$

38. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1.$

39. $x^3 + y^3 = 3xy.$

40. $xy - \sin x = 0.$

41. $xy - \log x = 0.$

42. $y^3 + 4x^2y + xy^2 - 4x + 8 = 0.$

In each of the following, find d^2y/dx^2:

43. $y = x^2 \log x.$

44. $y = x \arcsin x.$

45. $x^2 + y^2 = 4x.$

46. $x^3 + y^3 = 3x.$

47. $y = e^{-x^2}.$

48. $y = \sec^2 x.$

49. $x = 4 \cos \theta,\ y = 3 \sin \theta.$ (Use two methods.)

50. $x = \cot \theta,\ y = \sin^2 \theta.$ (Use two methods.)

51. Show that the tangents drawn to a parabola at the ends of the latus rectum are perpendicular to each other.

52. Write the equation of the tangent line to the parabola $y = x^2 - 7x + 10$ which is parallel to the line $y - 6 = 3x.$

53. Determine the angles of intersection of the line $y - x = 3$ and the curve $y = x^2 - 7x + 10.$

54. Determine the angles of intersection of the parabolas $x^2 - 4y = 0,$ and $y^2 = \frac{1}{2}x.$

55. Find the radius of curvature of the ellipse $x^2 + 4y^2 = 10x$ at (2, 2).

56. Find the inflection point on the curve $y = e^{\frac{1}{1-x}}$. Sketch the curve.

57. Determine maximum, minimum, and inflection points on the curve $y = \dfrac{4x}{x^2 + 4}$. Sketch the curve.

58. Show that the curve $y = x \log x$ is everywhere concave upward. Find the minimum point and sketch the curve.

59. Determine the minimum radius of curvature of $y = x^2 + 4x - 5$.

60. A glass jar of given volume is to be made with a metal top. The top costs twice as much per unit area as the glass. What should be the ratio of diameter to height for minimum cost?

61. The distances of the top and bottom of a picture above the level of the observer's eye are p and q, respectively. Show that the picture subtends a maximum angle at the eye if the observer stands back a distance $\sqrt{pq}$.

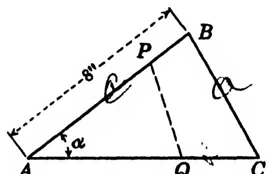

FIG. 63.

62. Locate points P and Q (Fig. 63) if area $APQ = \frac{1}{2}ABC$ and the length PQ is a minimum.

63. The sum of the length and the girth of a parcel-post package cannot exceed 100 in. What dimensions would give the greatest volume if the package is to be rectangular with a square cross section?

64. The crank arm OA (Fig. 64) rotates at a uniform angular velocity of 60 r.p.m. Express the linear velocity of the piston at B as a function of θ.

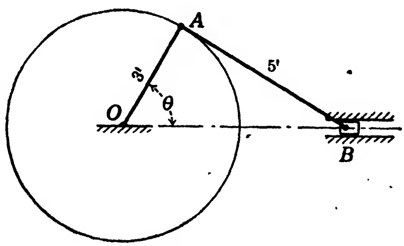

FIG. 64.

65. A kite is 120 ft. high and 200 ft. of string are out. If the kite is moving horizontally at 6 m.p.h., how fast is the string being paid out if the person flying the kite is walking in the same direction at 2 m.p.h.?

66. A line of length L has its ends on the positive x and y axes. For what position is the vertical distance from $P(a, 0)$ to the line a maximum? $a < L$.

67. The frictional resistance to flow of a liquid in an open channel is proportional to the area of the wetted surface. Show that for a rectangular channel the width should be twice the depth for minimum wetted area.

68. An open channel is to be constructed with trapezoidal cross section of area A. The inclination of the sides is a given angle α. Determine the shape which gives a minimum wetted area.

69. A fence h ft. high is parallel to, and b ft. distant from, the wall of a building. Show that the inclination of the shortest ladder which will reach to the wall from the ground outside the fence is given by $\tan^3 \theta = h/b$.

70. Prove that the area of the smallest ellipse which can be circumscribed about a rectangle of area A is independent of the shape of this rectangle and equal to $\pi A/2$.

CHAPTER XII

THE LIMIT OF A FUNCTION

(Continued from Chapter II)

61. Introduction.—We have defined the limit of a function $f(x)$ as $x \to a$ to be the constant L (if it exists) to which the value of $f(x)$ is arbitrarily near if x is sufficiently near to, but not equal to, a.

We have found that the problem of computing this limit is trivial if $f(x)$ is continuous at $x = a$; for we can find $f(a)$ by direct substitution and this is the desired limit. If however the substitution of a for x leads to a meaningless symbol,* (such as $0/0$), the function is undefined at $x = a$, and the problem of computing its limit as $x \to a$, if such limit exists, becomes more difficult.

It will be recalled that in Chap. II we considered the function

$$f(x) = \frac{x^2 - 16}{x - 4}$$

which takes on the form $0/0$ if we formally substitute 4 for x; the above relation therefore yields no value for $f(x)$ when $x = 4$. We were able to show however that if x is very *near* 4, the value of this function is very *near* 8; in fact, we showed that

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = 8.$$

Similarly the function

$$\varphi(x) = \frac{\sin x}{x}$$

* In such a case the value of the function is sometimes said to be *indeterminate*, or the function is said to take on an *indeterminate form*.

is undefined at $x = 0$; we proved, however, in Chap. II that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

We shall now discuss a theorem which is useful in connection with the problem of finding the limit of a function.

62. Lhopital's* Theorem.—*Suppose that two functions $g(x)$ and $h(x)$ both have the value 0 when $x = a$; then the limit of their quotient $g(x)/h(x)$ as x approaches a is equal to the limit of the quotient of their derivatives, if it exists.* That is, *if* $g(a) = h(a) = 0$, then

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{g'(x)}{h'(x)} \text{ (if the limit exists).}$$

The usefulness of this theorem lies of course in the fact that the latter limit may be easier to compute. Thus, in the example mentioned above we have

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{\dfrac{d}{dx}(x^2 - 16)}{\dfrac{d}{dx}(x - 4)}$$

$$= \lim_{x \to 4} \frac{2x}{1} = 8.$$

Similarly,

$$\lim_{x \to 0} \frac{3 \sin 4x}{2x} = \lim_{x \to 0} \frac{\dfrac{d}{dx}(3 \sin 4x)}{\dfrac{d}{dx}(2x)}$$

$$= \lim_{x \to 0} \frac{12 \cos 4x}{2}$$

$$= \frac{12}{2} = 6.$$

If the limit of the quotient $g'(x)/h'(x)$ does not exist, but instead $g'(x)/h'(x) \to \infty$ as $x \to a$, then the same is true of

* Also spelled L'Hospital. The spelling used, however, is preferred by Webster.

the original quotient. Thus,

$$\lim_{x\to 0}\frac{\tan x}{x^2} = \lim_{x\to 0}\frac{\dfrac{d}{dx}(\tan x)}{\dfrac{d}{dx}(x^2)}$$

$$= \lim_{x\to 0}\frac{\sec^2 x}{2x} = \infty.$$

In this last fraction the numerator approaches 1 and the denominator approaches 0 when $x \to 0$; the value of the fraction therefore increases beyond bound. The result means, roughly speaking, that when $x \to 0$, x^2 approaches

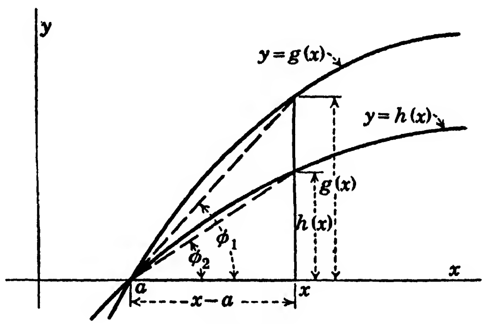

FIG. 65.

zero so much more rapidly than does tan x that the quotient becomes indefinitely large.

If we are seeking the limit of the fraction as $x \to \infty$ instead of as $x \to a$, and if it happens that both numerator and denominator approach 0 as $x \to \infty$, the same theorem applies. And finally, if the fraction takes symbolically the form ∞/∞ * instead of $0/0$ when $x \to a$ or when $x \to \infty$, the theorem again applies.

Example

Compute $\lim\limits_{x\to 0}\dfrac{\log x}{\cot x}$.

* The fraction $g(x)/h(x)$ is said to take the meaningless or "indeterminate" form ∞/∞ for $x = a$ if $\lim\limits_{x\to a} g(x) = \infty$ and $\lim\limits_{x\to a} h(x) = \infty$.

Solution

Both $\log x$ and $\cot x$ increase beyond bound in absolute value as $x \to 0$ and the fraction is therefore said to take the form ∞/∞ for $x = 0$. What happens to the value of the fraction as $x \to 0$ may be said, roughly speaking, to depend upon which one becomes large the more rapidly. Applying the theorem we have

$$\lim_{x\to 0} \frac{\log x}{\cot x} = \lim_{x\to 0} \frac{\dfrac{d}{dx}(\log x)}{\dfrac{d}{dx}(\cot x)}$$

$$= \lim_{x\to 0} \frac{\dfrac{1}{x}}{-\csc^2 x} = \lim_{x\to 0}\left(-\frac{\sin^2 x}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot \lim_{x\to 0}(-\sin x)$$

$$= 1 \cdot 0 = 0.$$

What does the result mean?

No attempt will be made here to specify all of the conditions which must be satisfied by the functions $g(x)$ and $h(x)$; neither will a complete proof of the theorem for the various cases be given. The following proof for the case in which $g(x)/h(x)$ takes the form $0/0$ for $x = a$ is however extremely simple. It is assumed that both $g(x)$ and $h(x)$ are continuous and differentiable in the neighborhood of $x = a$.

If $g(a) = h(a) = 0$, the graphs of $g(x)$ and $h(x)$ both cross (or touch) the x-axis at $x = a$ as indicated in Fig. 65. At a neighboring point the value of the quotient $g(x)/h(x)$ is *identically* the same as the quotient $\tan\varphi_1/\tan\varphi_2$; *i.e.*,

$$\frac{g(x)}{h(x)} \equiv \frac{\dfrac{g(x)}{x-a}}{\dfrac{h(x)}{x-a}} \equiv \frac{\tan\varphi_1}{\tan\varphi_2}.$$

Now letting $x \to a$ we have

$$\lim_{x\to a}\frac{g(x)}{h(x)} = \lim_{x\to a}\frac{\tan\varphi_1}{\tan\varphi_2} = \frac{g'(a)}{h'(a)} \qquad [\text{if } h'(a) \neq 0].$$

If the quotient $g'(x)/h'(x)$ also takes the form $0/0$ for $x = a$, the theorem may be applied in turn to this quotient; *i.e.*, we may examine the quotient of the second derivatives.

Example

Compute $\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x}$.

Solution

For $x = 0$ the fraction reduces to $0/0$; hence,

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x} = \lim_{x \to 0} \frac{\frac{d}{dx}(e^x + e^{-x} - 2\cos x)}{\frac{d}{dx}(x \sin x)}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + 2\sin x}{x \cos x + \sin x}.$$

But this new fraction also takes the form $0/0$ for $x = 0$; hence, we apply the theorem again obtaining

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(e^x - e^{-x} + 2\sin x)}{\frac{d}{dx}(x \cos x + \sin x)}$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x}{-x \sin x + 2\cos x}$$

$$= \tfrac{4}{2} = 2.$$

PROBLEMS

1. Is it obvious to you that if $g(x)$ and $h(x)$ both approach 0 as $x \to a$, the fraction $g(x)/h(x)$ may approach *any* number k (including 0) as a limit or may increase indefinitely? Discuss the behavior of the fractions x/x^2, x^2/x, and $3x/x$ as $x \to 0$.

2. Discuss Prob. 1 for the case in which $g(x)$ and $h(x) \to \infty$ as $x \to a$. Consider the same three fractions as $x \to \infty$.

3. Using Lhopital's Theorem show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Is this an acceptable proof or has the result been assumed?

4. Compute $\lim_{x \to 0} \frac{3 \tan 2x}{4x}$ both with and without the use of Lhopital's Theorem.

5. Compute $\lim_{x \to 3} \frac{x + 5}{x - 1}$. Note that Lhopital's Theorem does not apply. Why?

6. Show that if x is near 2, the value of $\dfrac{x^3 - 8}{x - 2}$ is near 12.

7. Show that if x is near 2, the value of $\dfrac{x^2 + 8x - 20}{2x^2 - 5x + 2}$ is near 4.

8. Discuss the behavior of the function $f(x) = \dfrac{x^3 - 2x^2 - 9}{x^2 - 4x + 3}$ for x in the neighborhood of 3.

9. Show that the value of $(e^x - e^{-x})$ is about twice that of $\sin x$ if x is near zero.

10. Compute $\lim\limits_{x \to 0} \dfrac{x^2 \cos x}{2 \sin^2 \frac{1}{2}x}$ by two methods.

11. Find $\lim\limits_{\theta \to 0} \dfrac{\sin \theta(1 - \cos \theta)}{\theta^3}$ both with and without using Lhopital's Theorem.

Compute the following limits if they exist:

12. $\lim\limits_{\theta \to 0} \dfrac{1 - \cos \theta}{\theta^2}$.

13. $\lim\limits_{\theta \to 0} \dfrac{\theta - \sin \theta}{\theta^3}$.

14. $\lim\limits_{x \to 1} \dfrac{\log x}{(x - 1)}$.

15. $\lim\limits_{x \to 0} \dfrac{\log \cos x}{x}$.

16. $\lim\limits_{\theta \to 0} \dfrac{\sin \theta(1 - \cos \theta)}{\theta^3 \cos \theta}$.

17. $\lim\limits_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x}$.

18. $\lim\limits_{x \to 0} \dfrac{12^x - 4^x}{x}$.

19. $\lim\limits_{x \to 0} \dfrac{\sin x - x}{\tan x - x}$.

20. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$.

21. $\lim\limits_{x \to 0} \dfrac{x\,(x - \sin x)}{e^x + e^{-x} - (x^2 + 2)}$.

22. $\lim\limits_{x \to \infty} \dfrac{2x^3 + 4x - 1}{4x^3 - 7x + 6}$.

23. $\lim\limits_{x \to \infty} \dfrac{4x^3 - 3}{x^2 + 4x + 2}$.

24. $\lim\limits_{x \to \infty} \dfrac{x^2 + 7x + 5}{6x^4 + 5x}$.

25. $\lim\limits_{x \to 0} \dfrac{\cot 2x}{\cot x}$.

26. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sec x}{\log \sec x}$.

27. $\lim\limits_{x \to 4} \dfrac{\log (x - 4)}{\cot \pi x}$.

28. $\lim\limits_{x \to \infty} \dfrac{e^x}{x^4}$.

29. Is the value of $2^x/x^{1,000,000}$ large or small for very large values of x?

30. Show that however large k may be, $\lim\limits_{x \to \infty} \dfrac{e^x}{x^k} = \infty$.

Determine each of the following limits by first writing the product in the form of a quotient and then applying the method used in the preceding problems:

31. $\lim\limits_{x \to \infty} x \sin \dfrac{2\pi}{x}$.

32. $\lim\limits_{x \to 0} \sin x \log x$.

33. $\lim\limits_{x \to 0} x \csc 4x$.

34. $\lim\limits_{x \to 1} (x - 1) \tan \dfrac{\pi x}{2}$.

CHAPTER XIII

THE DIFFERENTIAL

63. Infinitesimals.—In this chapter we shall use the word *infinitesimal* to denote a *variable quantity which is approaching zero as a limit.* We have already employed such quantities in connection with the definition of the derivative and the development of formulas for differentiation. Thus in defining the value of dy/dx at a point $A(x, y)$ on the curve shown in Fig. 66, our procedure was to form the quotient $\Delta y/\Delta x$ and then let $\Delta x \rightarrow 0$; thus, Δx plays the role of an infinitesimal. Since Δy approaches zero simultaneously with Δx, it is also an infinitesimal.

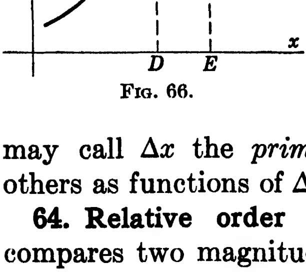

Fig. 66.

Other quantities that approach zero along with Δx are the length of chord AB, the area of triangle $ABC(=\frac{1}{2}\Delta x\,\Delta y)$, the area of rectangle $ACED(=y\,\Delta x)$, etc. We may call Δx the *primary infinitesimal* and regard these others as functions of Δx.

64. Relative order of infinitesimals.—One ordinarily compares two magnitudes by considering not their difference but their *ratio.* Thus, one might well say that two automobiles weighing 3,200 lb. and 3,220 lb. respectively are about equal in weight. He would not make the same remark about two dogs weighing respectively 5 lb. and 25 lb. The difference is 20 lb. in each case, but one is thinking of the ratio rather than the difference when he compares them.

Let α and β represent two infinitesimals, *i.e.*, two variable quantities which are simultaneously approaching zero as a

limit. If we wished to make a relative comparison of their magnitudes at any instant we would of course consider the value of their ratio at that instant. Consider for example the quantities x^2 and $5x$, both of which approach zero if $x \to 0$. Their ratio, when $x = 0.1$ is $0.01/0.5 = \frac{1}{50}$; when $x = 0.01$ it is $0.0001/0.05 = \frac{1}{500}$. It is obvious that their ratio approaches *zero* as $x \to 0$ and we say, under these conditions, that x^2 is an infinitesimal of *higher order* than $5x$. Its value becomes an arbitrarily small fraction of that of $5x$ when x is taken sufficiently near to (but not equal to) zero. In general we define the relative order of two infinitesimals α and β as follows:

If $\lim \frac{\alpha}{\beta} = 0$, α is of higher order than β.

If $\lim \frac{\alpha}{\beta} = k \neq 0$, α is of the same order as β.

If $\lim \frac{\alpha}{\beta} = \infty$, α is of lower order than β.

Example 1

Determine whether $\sin 2x$ is of the same order as $\sin x$ when $x \to 0$.

Solution

When $x \to 0$, both $\sin 2x$ and $\sin x$ approach zero; and

$$\lim_{x\to 0} \frac{\sin 2x}{\sin x} = \lim_{x\to 0} \frac{2\cos 2x}{\cos x} = 2.$$

The infinitesimals are then of the *same* order. The fact that the limit is 2 means that the value of $\sin 2x$ is practically twice that of $\sin x$ when x is near to but not equal to zero.

Example 2

Show that x^3 is an infinitesimal of higher order than x^2 when $x \to 0$.

Solution

$$\lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0.$$

x^3 is therefore of higher order than x^2 when $x \to 0$. Its value conse-

quently is an arbitrarily small fraction of that of x^2 when x is sufficiently near to but not equal to zero.

If α is of higher order than β, we may compare them somewhat more precisely as follows: *If there exists a positive integer n such that*

$$\lim \frac{\alpha}{\beta^n} = k \neq 0,$$

then α is said to be of the nth order relative to β.

Example

Show that when $x \to 0$ the difference between $\sin x$ and x is of third order relative to x.

Solution

$$\lim_{x \to 0} \frac{\sin x - x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} - \frac{x}{x}\right) = 1 - 1 = 0.$$

Therefore $(\sin x - x)$ is certainly of higher order than x. Next compare $(\sin x - x)$ with x^2:

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = \lim_{x \to 0} \frac{\cos x - 1}{2x}$$
$$= \lim_{x \to 0} \frac{-\sin x}{2} = 0.$$

Therefore $(\sin x - x)$ is of higher order than x^2. Next compare with x^3:

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$$
$$= \lim_{x \to 0} \frac{-\sin x}{6x}$$
$$= \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

Therefore $(\sin x - x)$ is of the same order as x^3 and is of third order relative to x.

65. The case in which $\lim \alpha/\beta = 1$.—Two infinitesimals are of the same order if the limit of their ratio is *any* constant $k \neq 0$. Thus $6x$ and $2x$ are of the same order since

$$\lim_{x \to 0} \frac{6x}{2x} = 3.$$

It should be observed that in this case their difference, $4x$, is not of higher order but is also of the *same* order.

It will now be shown that if the value of k is *one* the difference *is* of higher order; *i.e.*, if

$$\lim \frac{\alpha}{\beta} = 1$$

then $\alpha - \beta$ *is of higher order than* α *or* β. In order to prove this, we merely note that

$$\begin{aligned}\lim \frac{\alpha - \beta}{\beta} &= \lim \frac{\alpha}{\beta} - \lim \frac{\beta}{\beta}\\ &= \lim \frac{\alpha}{\beta} - 1\\ &= 0, \textit{ if and only if } \lim \frac{\alpha}{\beta} = 1.\end{aligned}$$

PROBLEMS

1. State which of the following functions of x are infinitesimals if $x \to 0$: $\sin x$, $\cos x$, e^x, x^3, $\log (1 + x)$.

2. Show that $\log (1 + x)$ is an infinitesimal of the same order as x when $x \to 0$.

3. Show that $\sin x$ and $\tan x$ are infinitesimals of the same order when $x \to 0$. What general statement can you make about their difference?

4. Show that the difference between $\tan x$ and x is of third order relative to x when $x \to 0$.

5. Show that the difference between $\cos x$ and 1 is of second order relative to x when $x \to 0$.

6. Show that the difference between $\sin x$ and $\tan x$ is of third order relative to x when $x \to 0$.

7. Show that the difference between $\sin 2x$ and $2 \sin x$ is of higher order than either when $x \to 0$. What is the order of this difference?

8. Show that if α' and β' differ respectively from α and β by infinitesimals of higher order, then

$$\lim \frac{\alpha'}{\beta'} = \lim \frac{\alpha}{\beta}.$$

HINT: $\frac{\alpha'}{\beta'} = \frac{\alpha}{\beta}\left[\frac{\alpha'}{\alpha} \cdot \frac{\beta}{\beta'}\right]$.

9. Why is Δy in general an infinitesimal of the same order as Δx (Fig. 66)? What does $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ represent geometrically?

10. Why is the chord AB an infinitesimal of the same order as Δx (Fig. 66)? What does $\lim\limits_{\Delta x \to 0} \dfrac{\overline{AB}}{\Delta x}$ represent?

11. Show that the area of triangle ABC (Fig. 66) is an infinitesimal of higher order than that of rectangle $ACED$ when $\Delta x \to 0$. Show in particular that it is of second order.

12. Show that the area bounded by the arc AB, the two ordinates and the x-axis (Fig. 66) is more than $y\,\Delta x$ and less than $(y + \Delta y)\Delta x$; hence, show that this area is represented by $y\,\Delta x$ except for an infinitesimal of higher order than $y\,\Delta x$.

66. The differential of a function.—Consider the function $y = f(x)$ whose graph is shown in Fig. 67. Starting at any point $A(x, y)$ let x increase by a small amount Δx; the corresponding increment in y is represented by $CB = \Delta y$.

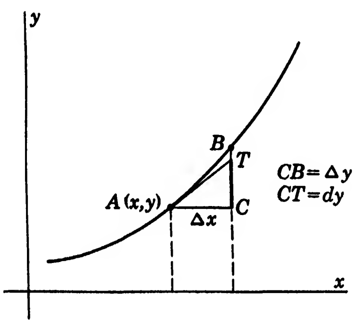

FIG. 67.

To obtain an approximation to Δy let us multiply the slope of the tangent at A by Δx obtaining

$$CT = f'(x)\Delta x.$$

It is obvious that if Δx is small this is a good approximation to Δy. In fact the difference between this and Δy is an infinitesimal of higher order. For

$$\lim_{\Delta x \to 0} \frac{\Delta y - f'(x)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} - f'(x) = 0.$$

The increment Δy may therefore be regarded as consisting of two parts:

1. The part represented by $CT = f'(x)\Delta x$; this is the larger part* and is called the *principal part* of the increment.

2. The comparatively small part represented by TB which is of higher order; this part is an arbitrarily small fraction of Δy when Δx is sufficiently small.

The principal part of the increment of a function $y = f(x)$ is called the *differential* of the function and is denoted by

* At least when Δx is sufficiently small.

the symbol dy; *i.e.*,

$$dy = f'(x)\Delta x.$$

If we take this as the definition of the differential of *any* function and apply it to the particular function $f(x) = x$ we have, since $f'(x) = 1$,

$$dx = 1\,\Delta x;$$

i.e., the differential of the independent variable is equal to its increment. We may then write our definition of dy in the form

$$dy = f'(x)dx.$$

Up to now we have not regarded the symbol dy/dx as a fraction; the separate symbols dy and dx had no meanings. With the above definition it is now clear that we may, if we choose, think of dy/dx as an ordinary fraction; *i.e.*, (Fig. 67)

$$\frac{dy}{dx} = \frac{\text{the principal part of the increment in } y}{\text{the principal part of the increment in } x} = \frac{CT}{AC}.$$

If x is the independent variable, the principal part of its increment is the whole increment. If, however, y is defined as a function of x by the parametric equations

$$y = g(t) \qquad x = h(t)$$

in which t is considered the independent variable, then

$$dt = \Delta t; \qquad dy = g'(t)dt; \qquad dx = h'(t)dt.$$

67. Method of obtaining the differential.—From the definition it is evident that in order to obtain the differential dy of a function $y = f(x)$ it is only necessary to multiply the derivative by dx.

***Example* 1**

$$y = 6x^2 + 3;$$

$$\frac{dy}{dx} = 12x$$

$$dy = 12x\,dx.$$

Example 2

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$
$$dA = 2\pi r\, dr.$$

68. Application to approximate formulas.—An immediate application of the differential of a function comes from the fact that if Δx is small, dy is a good approximation to Δy.

Example 1

Compute approximately the volume of metal in a hollow spherical shell, the thickness being 0.05 in. and the inside radius 5 in.

Solution

The volume of metal is equal to the amount by which the volume of a sphere increases when its radius changes from 5 to 5.05 in. Using dv as an approximation to this we have

$$v = \tfrac{4}{3}\pi r^3;$$
$$\frac{dv}{dr} = 4\pi r^2$$
$$dv = 4\pi r^2\, dr.$$

Taking $r = 5$ in. and $dr = 0.05$ in., we find

$$dv = 4\pi(25)(0.05) = 15.7 \text{ cu. in.}$$

By the usual methods one finds the actual volume to be 15.87 cu. in.

Example 2

Compute approximately $(1.98)^5$.

Solution

We may find the amount by which x^5 decreases when x decreases from 2 to 1.98 and subtract this from 32.

$$y = x^5$$
$$dy = 5x^4 dx.$$

Taking $x = 2$ and $dx = -0.02$, we find

$$dy = 5(16)(-0.02) = -1.6;$$

i.e., x^5 decreases by 1.6 when x changes from 2 to 1.98. Hence

$$(1.98)^5 = 32 - 1.6 = 30.4.$$

Using logarithms, one obtains the value 30.43.

PROBLEMS

1. Sketch the curve $v = \frac{4}{3}\pi r^3$.

(*a*) What ordinate represents v if $r = 5$ in.? 5.05 in.?

(*b*) What length represents the volume of a shell with radii 5 and 5.05 in.?

(*c*) What length represents the approximation to this volume which was found in Example 1, page 136.

2. Sketch the curve $y = x^5$.

(*a*) What ordinate represents 2^5? 1.98^5?

(*b*) What length represents the approximation which was found in Example 2, page 136?

3. Compute approximately the area of a walk 3 ft. wide around a city square which is 100 yd. on a side excluding the walk. Make a sketch of the square and walk and indicate the part of the walk which is neglected in the approximation.

4. Derive an approximate formula for the volume of metal in a hollow cylindrical pipe of length L and inside diameter D, the thickness t of the wall being small in comparison with D.

5. The radius of a circle is 20 in. Compute approximately the decrease in area when the radius decreases by 0.1 in.

6. Derive an approximate formula for the volume of metal in a can having the shape of a cube of edge x, the thickness t of the metal being small.

7. Compute approximately the value of tan 46°. Illustrate graphically. HINT: Find the increase in $\tan x$ when x increases from 45 to 46° and add this to 1. Why? $1° = 0.0175$ radian.

8. Compute approximately the value of cos 62°.

9. Sketch the curve $y = 1/x$. Show that a small change Δx in x produces a change of approximately $-\frac{\Delta x}{x^2}$ in y. Use this fact to compute $\frac{1}{102}$.

10. Compute approximately $\log_{10} 102$. Illustrate with a graph.

11. Compute approximately $\sqrt[3]{28}$ using the differential. Show that the process is equivalent to expanding $(27 + 1)^{\frac{1}{3}}$ by the binomial theorem and using only the first two terms.

12. Compute approximately $\sqrt[5]{34}$. Why would this same procedure give a less accurate result when used to find $\sqrt[5]{40}$?

Make the following calculations approximately using differentials. Illustrate each with a graph.

13. $\sqrt{390}$. **14.** $(2.97)^4$.

15. $\sqrt[3]{130}$. **16.** $(2.02)^6$.

17. $\sqrt[5]{240}$. **18.** $3^{2.05}$.

19. sin 28°. **20.** tan 44°.

21. By dividing the differential of y by the differential of x, find dy/dx for the cycloid

$$y = a(1 - \cos\theta), \qquad x = a(\theta - \sin\theta).$$

22. By dividing the differential of y by the differential of x, find dy/dx for the curve

$$y = a(2\sin t - \sin 2t), \qquad x = a(2\cos t - \cos 2t).$$

23. Find the derivative of the volume of a sphere with respect to its surface area by dividing the differential of the volume by that of the surface area, the radius being regarded as the independent variable.

24. Find the derivative of the volume of a cube with respect to its surface area by the method indicated in Prob. 23.

69. Application to small errors.—In scientific work one often determines the value of a quantity x by measurement and uses this measured value in computing the value of another quantity y. From a knowledge of the instruments used he may be able to estimate the greatest possible error in the measurement of x. He is then interested in knowing the corresponding possible error in y.

Example

The diameter of a cylindrical bar is found to be 4.2 in. with a possible error of 0.05 in. What is the greatest possible error in the computed area?

Solution

The true diameter is between 4.15 and 4.25 in. The greatest possible error is then the amount by which the area of a circle increases when its diameter changes from 4.2 to 4.25 in. Using dA as an approximation to this we have

$$A = \frac{\pi D^2}{4}$$

$$dA = \frac{\pi D}{2}dD,$$

Taking

$$D = 4.2 \qquad \text{and} \qquad dD = 0.05,$$

we find

$$dA = \pi(2.1)(0.05) = 0.33 \text{ sq. in.}$$

The *possible relative error* may be defined as the ratio of the maximum possible error to the computed value. In the above case the possible relative error is

$$\frac{dA}{A} = \frac{0.33}{\pi(2.1)^2} = \frac{0.33}{13.85} = 0.024, \text{ or } 2.4 \text{ per cent.}$$

It is possible to find the relative error dA/A directly by taking the natural logarithm of each side before differentiating; thus

$$A = \frac{\pi D^2}{4}.$$

$$\log A = \log\frac{\pi}{4} + 2\log D$$

$$\frac{1}{A}\frac{dA}{dD} = \frac{2}{D}$$

$$\frac{dA}{A} = 2\frac{dD}{D}.$$

This equation states that the relative error in A is twice the relative error in D; taking $D = 4.2$ and $dD = 0.05$ we have

$$\frac{dA}{A} = 2\frac{0.05}{4.2} = 0.024, \text{ or } 2.4 \text{ per cent.}$$

PROBLEMS

1. A square lot with sides 80 ft. long by measurement is laid out. If there is a possible error of 3 in. in each side, what is approximately the greatest error in the computed area?

2. A triangle is found by measurement to be equilateral with sides 60 ft. long. If there is a possible error of 3 in. in each side, what is approximately the greatest error in the computed area?

3. The edges of a cube are, by measurement, 10 in. long. If there is a possible error of 0.05 in. in each edge, what is approximately the greatest error in the computed area?

4. A cubical box is to be made to hold 1,000 cu. in. What is the allowable error in the edge if the volume must be accurate to 10 cu. in.?

5. A 12-lb. shot is to be made of iron weighing 450 lb. per cubic foot. If the weight must be accurate to 1 oz., what should be its radius and what is the allowable error?

6. Suppose that the radius of a sphere can be measured with an error not more than 0.001 in. For what size spheres would this give an error less than 1 cu. in. in the computed volume?

7. If an angle θ can be measured with an error not more than $\frac{1}{4}°$, for what size angles will the error in the value of $\tan\theta$ be less than 0.1?

8. What is approximately the error in $\cos\theta$ due to a small error $d\theta$ in measuring θ? Explain the negative sign.

9. Show that the error in $\tan\theta$ due to the same small error $d\theta$ in measuring θ is approximately three times as large when θ is near 60° as when it is near 30°.

10. Show that an error of 1 per cent in measuring the radius of a sphere will result in an error of approximately 3 per cent in the computed volume.

11. Find the approximate relative error in the computed volume and surface area of a cube due to an error of 1 per cent in the edge.

12. Show that the relative error in the computed value of a function of the form kx^n, due to a small error dx in x, is approximately n times the relative error in x.

13. The period of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. By what per cent should the length be changed to correct for a loss of 2 min. per day?

14. Show that if θ is near $\pi/4$ the relative error in $\tan\theta$, due to a small error $d\theta$ in measuring θ, is approximately $\pi/2$ times the relative error in θ.

CHAPTER XIV

INTEGRATION

70. Introduction.—Up to now we have been concerned with the problem: Given a function $f(x)$, to obtain its derivative. In many important applications of the calculus we are confronted by the inverse problem, namely: Given the derivative $f'(x)$ or the differential $f'(x)dx$ of a function, to find the function. We shall see later that the problems of finding the length of a curve, the area bounded by a curve, the work done by a variable force etc., all lead to this inverse problem.

Suppose, for example, that we wished to find the length of the arc of the parabola $y = x^2$ from (0, 0) to (2, 4). Regarding the length s of the arc from (0, 0) to any other point (x, y) on the curve as a function of x, we know from the formula for the derivative of arc that

$$\frac{ds}{dx} = \sqrt{1 + 4x^2}.$$

The first step in solving our problem would, therefore, be to find a function

$$s = F(x)$$

whose derivative is $\sqrt{1 + 4x^2}$, or whose differential is $\sqrt{1 + 4x^2}\,dx$. Such a function is called an *integral* of $\sqrt{1 + 4x^2}\,dx$ and the process of finding it is called *integration*. It is denoted by the symbol

$$\int(\quad)dx$$

which thus denotes the inverse of the operation which was denoted by the symbol

$$\frac{d}{dx}(\quad).$$

Thus, just as we use the abbreviation $\frac{d}{dx}(x^2) = ?$ to mean, "what is the derivative with respect to x of x^2?", we use the abbreviation

$$\int x^2 dx = ?$$

to mean, "what is the function whose derivative is x^2—or whose differential is x^2dx?" The function which is to be integrated (x^2 in this case) is called the *integrand.*

The present chapter will be devoted to the problem of gaining some facility in the technique of integration.

71. The problem of integration.—The definition of the derivative carried with it a formal process by which one could find the derivative of a given function. There is no process for integration which quite corresponds to this. Given a function to be integrated one tries to determine, from his knowledge of differentiation, what function or what type of function might have this given function for its derivative. Thus, one finds easily that

$$\int \cos x \, dx = \sin x$$

since the derivative of $\sin x$ is known to be $\cos x$. Almost as easily one finds that

$$\int x^2 dx = \frac{x^3}{3}$$

since the derivative of $x^3/3$ is x^2. In the case of

$$\int \sqrt{1 + 4x^2}\, dx,$$

the result cannot be obtained by inspection; *i.e.*, one cannot think of any simple function whose derivative is $\sqrt{1 + 4x^2}$. The result, which is obtained by methods to be explained later, is

$$\int \sqrt{1 + 4x^2}\, dx = \frac{x\sqrt{1 + 4x^2}}{2} + \frac{\log(\sqrt{1 + 4x^2} + 2x)}{4}.$$

The student may verify this integration by differentiating the right-hand side.

There is of course no assurance that an arbitrarily selected function can be integrated at all. Thus, while the integrand in

$$\int \sqrt{\sin x}\, dx$$

appears to be quite simple, it happens that there is no finite combination of the elementary functions which we have studied whose derivative is $\sqrt{\sin x}$. The integration cannot therefore be performed in the elementary sense.

72. The constant of integration.—Since the derivative of $x^3/3$ is x^2, it follows that $x^3/3$ is *an* integral of x^2dx. It is obvious, however, that

$$\frac{x^3}{3} + 4 \qquad \text{or} \qquad \frac{x^3}{3} - 7 \qquad \text{or} \qquad \frac{x^3}{3} + C$$

where C is *any* constant is also an integral, since the derivative is x^2 in each case. This last integral is the *most general function whose derivative is* x^2, in the sense that there is no function whose derivative is x^2 except those which can be obtained from it by giving particular values to C. It is called the *indefinite integral* and C is called the *constant of integration.* Integrals which may be obtained by giving particular values to C are called *particular integrals.*

That the indefinite integral obtained by adding an arbitrary constant C to any particular integral has the property just mentioned may be shown by proving that if $F(x)$ and $G(x)$ are two functions having the same derivative, they can differ only by a constant. Thus, if

$$\frac{d}{dx}F(x) = \varphi(x) \qquad \text{and} \qquad \frac{d}{dx}G(x) = \varphi(x)$$

then

$$\frac{d}{dx}[F(x) - G(x)] = 0.$$

Since the only function whose derivative is 0 for all values of x is a constant, we have

$$F(x) - G(x) = k.$$

73. Formulas for integration.—From the known formulas for differentiation, we can write down immediately a list of formulas for integration. They can be easily memorized by thinking of the corresponding formulas for differentiation.

(1) $$\int (du + dv + dw) = \int du + \int dv + \int dw.$$

(2) $$\int a\,dv = a \int dv.$$

(3) $$\int v^n dv = \frac{v^{n+1}}{n+1} + C. \qquad (n \neq -1)$$

(4) $$\int \frac{dv}{v} = \log v + C.$$

(5) $$\int a^v dv = \frac{a^v}{\log a} + C.$$

(6) $$\int e^v dv = e^v + C.$$

(7) $$\int \sin v\,dv = -\cos v + C.$$

(8) $$\int \cos v\,dv = \sin v + C.$$

(9) $$\int \sec^2 v\,dv = \tan v + C.$$

(10) $$\int \csc^2 v\,dv = -\cot v + C.$$

(11) $$\int \sec v \tan v\,dv = \sec v + C.$$

(12) $$\int \csc v \cot v\,dv = -\csc v + C.$$

(13) $$\int \frac{dv}{v^2 + a^2} = \frac{1}{a} \arctan \frac{v}{a} + C.$$

(14) $$\int \frac{dv}{\sqrt{a^2 - v^2}} = \arcsin \frac{v}{a} + C.$$

There are four additional formulas which the student will find convenient. They are not obtained from differentiation formulas and hence are more difficult to remember. The direct derivation of these will be assigned later as problems.

(15). $$\int \frac{dv}{v^2 - a^2} = \frac{1}{2a} \log \frac{v-a}{v+a} + C.$$

(16) $$\int \frac{dv}{\sqrt{v^2 \pm a^2}} = \log (v + \sqrt{v^2 \pm a^2}) + C.$$

(17). $$\int \sec v\, dv = \log (\sec v + \tan v) + C.$$

(18). $$\int \csc v\, dv = \log (\csc v - \cot v) + C.$$

74. Formulas (1) to (4).—It is extremely important that the student understand clearly the meaning of the formulas. Formula (1) states that the integral of the sum of several functions is equal to the sum of their separate integrals. Thus,

$$\int (x^3 + 4x^2 - 3)dx = \int x^3dx + \int 4x^2dx - \int 3\, dx.$$

Formula (2) states that the integral of the product of a constant and a function is equal to the constant times the integral of the function. Thus, in the above example,

$$\int 4x^2dx = 4\int x^2dx = 4 \cdot \frac{x^3}{3}.$$

$$\int 3\, dx = 3\int dx = 3x.$$

Finally, then,

$$\int (x^3 + 4x^2 - 3)dx = \int x^3dx + 4\int x^2dx - 3\int dx$$
$$= \frac{x^4}{4} + \frac{4}{3}x^3 - 3x + C.$$

Great care must be exercised in applying formulas (3) and (4). Formula (3) states that the integral of the product

of the nth power of a function v and **the differential dv of this function,** is equal to the $(n + 1)$th power divided by $(n + 1)$. Thus,

$$\int (x^2 + 4x + 1)^3(2x + 4)dx = \frac{(x^2 + 4x + 1)^4}{4} + C.$$

Here

$$v = x^2 + 4x + 1, \qquad dv = (2x + 4)dx.$$

The formula does not say, for example, that

$$\int (4x^2 + 1)^2 dx = \frac{(4x^2 + 1)^3}{3} + C.$$

This integrand is not in the form $v^n dv$ because, if it were, $v = 4x^2 + 1$, and differentiating $dv = 8x\,dx$. Since we do not have $8x\,dx$ the formula does not apply. The problem can be solved however by squaring out the integrand and then integrating. Thus,

$$\begin{aligned}\int (4x^2 + 1)^2 dx &= \int (16x^4 + 8x^2 + 1)dx \\ &= \frac{16x^5}{5} + \frac{8x^3}{3} + x + C.\end{aligned}$$

If only a *constant factor* is lacking, this can be easily supplied. Consider, for example,

$$\int x^2\sqrt{2x^3 + 9}\, dx.$$

If we let $v = 2x^3 + 9$, then $n = \frac{1}{2}$ and dv is $6x^2dx$. The above may be written in the form

$$\int (2x^3 + 9)^{\frac{1}{2}}(x^2dx)$$

where only the *constant* 6 is lacking. We may now *without changing the value of the integrand,* multiply by 6/6 and we have

$$\int (2x^3 + 9)^{\frac{1}{2}}\frac{6x^2dx}{6}.$$

The 6 in the denominator may be taken outside, leaving the integrand exactly in the form $v^n dv$. We have, then,

finally

$$
\begin{aligned}
\int x^2\sqrt{2x^3+9}\,dx &= \int (2x^3+9)^{\frac{1}{2}}x^2dx \\
&= \tfrac{1}{6}\int (2x^3+9)^{\frac{1}{2}}6x^2dx \\
&= \frac{1}{6}\frac{(2x^3+9)^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= \frac{1}{9}(2x^3+9)^{\frac{3}{2}} + C.
\end{aligned}
$$

As an example of the use of formula (4) consider

$$\int \frac{3x\,dx}{5-4x^2}.$$

We may let

$$v = 5 - 4x^2,$$

and differentiating

$$dv = -8x\,dx,$$

hence,

$$
\begin{aligned}
\int \frac{3x\,dx}{5-4x^2} &= 3\int \frac{x\,dx}{5-4x^2} \\
&= -\frac{3}{8}\int \frac{-8x\,dx}{5-4x^2} \\
&= -\frac{3}{8}\log(5-4x^2) + C.
\end{aligned}
$$

PROBLEMS

Integrate each of the following by reducing to the form $\int v^n dv$:

1. $\int (3x^4 + 4x^2 + 2)dx.$
2. $\int \sqrt{2x}\,dx.$
3. $\int \sqrt[3]{4x+1}\,dx.$
4. $\int \frac{dx}{x^3}.$
5. $\int 6t^{-\frac{3}{2}}dt.$
6. $\int \frac{y^4+1}{\sqrt{y}}dy.$
7. $\int 2s\sqrt{s}\,ds.$
8. $\int \sqrt{a^2-x^2}\,x\,dx.$
9. $\int (4 + x + x^2)^2dx.$
10. $\int \sin x \cos x\,dx.$

11. $\int \tan^2 3x \sec^2 3x\, dx.$

12. $\int \frac{\log^2 x\, dx}{x}.$

13. $\int \frac{2\, dx}{x^2 - 6x + 9}.$

14. $\int (a^{\frac{1}{2}} - x^{\frac{1}{2}})^2 dx.$

15. $\int \frac{e^x dx}{\sqrt{e^x + 4}}.$

16. $\int \frac{3 \arctan x\, dx}{x^2 + 1}.$

17. $\int \frac{(x^2 + 2)^3}{x^2} dx.$

18. $\int (1 + x)^2 \sqrt{x}\, dx.$

19. $\int (4 + x)(3 - x) dx.$

20. $\int \left(\frac{1}{x^2} - \sqrt[3]{8x} + \sqrt{x}\right) dx.$

Integrate the following by reducing to the form $\int \frac{dv}{v}$:

21. $\int \frac{dx}{2x + 1}.$

22. $\int \frac{x\, dx}{3x^2 - 1}.$

23. $\int \frac{\sec^2 \theta\, d\theta}{3 \tan \theta + 6}.$

24. $\int \tan x\, dx.$

HINT: Tan $x = \frac{\sin x}{\cos x}.$

25. $\int \cot x\, dx.$

26. $\int \frac{(x - 3) dx}{x^2 - 6x + 4}.$

27. $\int \frac{\sin \frac{1}{2}x\, dx}{4 \cos \frac{1}{2}x + 3}.$

28. $\int \frac{2^x\, dx}{2^x + 3}.$

29. $\int \frac{e^x dx}{3e^x + 4}.$

30. $\int \frac{ds}{s \log s}.$

Integrate the following using formulas (1) to (4):

31. $\int \frac{3x\, dx}{(4 - 2x^2)^2}.$

32. $\int (x^2 + 2)^2 x^2\, dx.$

33. $\int \frac{x\, dx}{\sqrt{a^2 - x^2}}.$

34. $\int \frac{\sec^2 \theta\, d\theta}{(1 + \tan \theta)^3}.$

35. $\int \frac{\csc^2 2x\, dx}{3 + 4 \cot 2x}.$

36. $\int \sin 2x \sin x\, dx.$

HINT: Sin $2x = 2 \sin x \cos x.$

37. $\int \cot \theta \log \sin \theta\, d\theta.$

38. $\int \frac{\log \tan \frac{1}{2}\theta}{\sin \theta} d\theta.$

39. $\int \sec \theta\, d\theta.$ *Formula* (17)

HINT: Multiply and divide by $(\sec \theta + \tan \theta)$.

40. $\int \csc \theta\, d\theta.$ *Formula* (18)

41. Evaluate $\int (x + 1)^2 dx$ both with and without expanding the integrand. Are the results the same?

42. Show that $\int \sin x \cos x\, dx = \frac{1}{2} \sin^2 x + C$ and that $\int \cos x \sin x\, dx = -\frac{1}{2} \cos^2 x + C$. Explain the apparent difference in the two results.

43. Explain why formula (3) cannot be used to evaluate

$$\int (x^2 + 4)^{\frac{1}{2}} dx.$$

44. Explain why formula (3) but not formula (4) could be used to evaluate $\int \frac{x\,dx}{\sqrt{4 - x^2}}$.

45. Explain why formula (3) cannot be used directly to evaluate $\int \cos^3 x\, dx$. Perform the integration using the fact that

$$\cos^2 x = 1 - \sin^2 x.$$

Could this same procedure be used on $\int \cos^4 x\, dx$?

75. Formulas (5) to (12), (17), (18).—Again the student is warned that he must exercise great care in fitting the given problem to the formula to be used. For example, in using formula (5) he must make certain that the quantity used for *dv is actually the differential of the function v* which is the exponent. Constant factors can of course be supplied as before. Thus,

$$\int 4^{3x+2} dx = \frac{1}{3} \int 4^{3x+2} 3\, dx = \frac{4^{3x+2}}{3 \log 4} + C.$$

Formula (6) could not be used to evaluate

$$\int e^{x^2} dx$$

since, if this were to be $e^v dv$, then

$$v = x^2 \qquad \text{and} \qquad dv = 2x\, dx.$$

PROBLEMS

Integrate the following by reducing to the form $\int a^v dv$ or $\int e^v dv$:

1. $\int e^{2x} dx.$

2. $\int 4^{3x+5} dx.$

3. $\int 6e^{\tan 2\theta} \sec^2 2\theta\, d\theta.$

4. $\int x e^{x^2} dx.$

5. $\int a^x e^x dx.$

6. $\int 4x^2 e^{-x^3} dx.$

7. $\int (e^{\frac{x}{2}} + e^{-\frac{x}{2}})dx.$

8. $\int (e^x + e^{-x})^2 dx.$

9. $\int \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})^2 dx.$

10. $\int \frac{e^{\log x^2} dx}{x}.$

11. $\int \frac{10^{\frac{1}{x^2}} dx}{x^3}.$

12. $\int e^{\sin x \cos x} \cos 2x \, dx.$

13. $\int e^{(\sin x + \cos x)^2} \cos 2x \, dx.$

14. $\int \frac{2^{\arctan x} dx}{x^2 + 1}.$

15. $\int (4x^2 e^{-x^3} + e^{-x})dx.$

Integrate the following using formulas (1) to (4), (7) to (12), (17), (18):

16. $\int \sin 4x \, dx = \frac{1}{4}\int \sin 4x \cdot 4 \, dx = -\frac{1}{4} \cos 4x + C.$

17. $\int \cos 2x \, dx.$

18. $\int \sec^2 (4x + 2)dx.$

19. $\int x \csc^2 (x^2 + 1)dx.$

20. $\int \frac{dx}{\sec \frac{1}{2}x}.$

21. $\int \sec 3\theta \tan 3\theta \, d\theta.$

22. $\int \frac{dt}{3 \cos^2 2t}.$

23. $\int \cot \frac{1}{2}x \sin \frac{1}{2}x \, dx.$

24. $\int 3 \sin \pi x \, dx.$

25. $\int (\sin x + \cos x)^2 dx.$

26. $\int \sec (3x + 4)dx.$

27. $\int x \csc x^2 dx.$

28. $\int \frac{dx}{\sin 3x}.$

29. $\int \frac{d\theta}{\cos \pi\theta}.$

30. $\int \frac{\tan \theta \, d\theta}{\sin 2\theta}.$

31. $\int \frac{\cos 2x \, dx}{\cos x}.$

32. $\int \frac{\cos x \, dx}{\sin \frac{1}{2}x}.$

33. $\int \frac{\sin 2\theta \, d\theta}{\cos^2 \theta \sin \theta}.$

34. $\int \frac{\tan x}{\cos x} dx.$

35. $\int 4 \sin 2x \sec x \, dx.$

36. $\int \frac{2 \cos x \, dx}{\sin^2 x}.$

37. $\int \sin \pi x \cos \pi x \, dx.$

38. Evaluate $\int \sin 2x \, dx$ using formula (7) and also by writing the integrand as $2 \sin x \cos x$. Explain the apparent difference in the results.

39. $\int \cos 2\theta \sin \theta \, d\theta.$

40. $\int \sec^2 \theta \tan 2\theta \, d\theta.$

76. Formulas (13) to (16).—In applying these formulas the student must again exercise great care. In using

formula (13) for example, he must make certain that the *numerator* of the integrand is *exactly the differential of the function whose square occurs in the denominator.* Constant factors can be supplied as before. Thus, we may fit

$$\int \frac{dx}{(3x+1)^2+6}$$

to formula (13), letting

$$v = 3x + 1 \qquad \text{and} \qquad a = \sqrt{6}.$$

Of course $dv = 3\,dx$ and we then have

$$\int \frac{dx}{(3x+1)^2+6} = \frac{1}{3}\int \frac{3\,dx}{(3x+1)^2+\sqrt{6}^2}$$
$$= \frac{1}{3}\left(\frac{1}{\sqrt{6}} \text{ arc tan } \frac{3x+1}{\sqrt{6}}\right) + C.$$

PROBLEMS

1. $\displaystyle\int \frac{dx}{9x^2+4}.$

2. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}.$

3. $\displaystyle\int \frac{dx}{\sqrt{10-4x^2}}.$

4. $\displaystyle\int \frac{x\,dx}{x^4-16}.$

5. $\displaystyle\int \frac{\sin x\,dx}{\sqrt{\cos^2 x+16}}.$

6. $\displaystyle\int \frac{e^x dx}{e^{2x}+4}.$

7. $\displaystyle\int \frac{dx}{\sqrt{a^2-b^2x^2}}.$

8. $\displaystyle\int \frac{dx}{x(\log^2 x+9)}.$

9. $\displaystyle\int \frac{t\,dt}{(t^2+4)^2-9}.$

10. $\displaystyle\int \frac{dx}{\sqrt{10-(1+2x)^2}}.$

11. $\displaystyle\int \frac{dx}{x^2+2x+5}.$

12. $\displaystyle\int \frac{dt}{4t^2-4t+7}.$

HINT: $x^2 + 2x + 5 = (x+1)^2 + 4$.

13. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}.$

14. $\displaystyle\int \frac{ds}{\sqrt{6s-s^2-5}}.$

HINT: $2x - x^2 = 1 - (1-x)^2$.

15. $\displaystyle\int \frac{x\,dx}{x^4+4x^2+14}.$

16. $\displaystyle\int \frac{dt}{4t^2-4t-3}.$

17. $\displaystyle\int \frac{t\,dt}{\sqrt{23+12t^2-4t^4}}.$

18. $\displaystyle\int \frac{dz}{\sqrt{z^2+2z+7}}.$

19. $\displaystyle\int \frac{2^x dx}{\sqrt{9 - 4^x}}$.

20. $\displaystyle\int \frac{dx}{\sqrt{16x^2 - 24x + 7}}$.

21. $\displaystyle\int \frac{\sec^2 u\, du}{\tan^2 u + 1}$ by two methods.

22. $\displaystyle\int \frac{e^{\frac{x}{2}} dx}{e^x - 1}$.

23. $\displaystyle\int \frac{(2x + 6)dx}{x^2 + 4x + 8}$.

HINT: Write $2x + 6$ as $(2x + 4) + 2$ and separate into two integrals.

24. $\displaystyle\int \frac{(8x - 1)dx}{4x^2 - 4x - 3}$.

25. $\displaystyle\int \frac{(18x - 3)dx}{\sqrt{9x^2 - 12x}}$.

26.
$$\int \frac{(2x + 1)dx}{4x^2 + 12x + 13} = \frac{1}{4}\int \frac{(8x + 4)dx}{4x^2 + 12x + 13}$$
$$= \frac{1}{4}\int \frac{(8x + 12 - 8)dx}{4x^2 + 12x + 13}$$
$$= \frac{1}{4}\int \frac{(8x + 12)dx}{4x^2 + 12x + 13} - 2\int \frac{dx}{4x^2 + 12x + 13}.$$

27. $\displaystyle\int \frac{(x + 3)dx}{x^2 + 4x - 5}$.

28. $\displaystyle\int \frac{(3x + 5)dx}{x^2 + x + 1}$.

29. $\displaystyle\int \frac{(2x - 7)dx}{\sqrt{3 - 6x - 9x^2}}$.

30. $\displaystyle\int \frac{(4x + 1)dx}{\sqrt{4x^2 - 12x + 5}}$.

31. $\displaystyle\int \frac{(4x + 7)dx}{\sqrt{x^2 - 3x + 2}}$.

32. $\displaystyle\int \frac{(x + 6)dx}{\sqrt{x^2 - 5x + 1}}$.

33. $\displaystyle\int \frac{(7x + 2)dx}{15 + 6x - 9x^2}$.

34. $\displaystyle\int \frac{(ax + b)dx}{cx^2 + dx + e}$.

77. Miscellaneous problems.—Success in integrating depends largely on one's ability to pick out the formula or procedure which is most apt to yield the desired result. Solving a large number of problems will give the student some facility in doing this.

PROBLEMS

1. $\displaystyle\int \frac{dx}{\sin x \tan x}$.

2. $\displaystyle\int \frac{\sin x\, dx}{\cos^2 x}$.

3. $\displaystyle\int (1 + \tan \theta)^2 d\theta$.

4. $\displaystyle\int \frac{(x + 2)^3 dx}{x^2}$.

5. $\displaystyle\int x(1 - \sqrt{x})^2 dx$.

6. $\displaystyle\int \sin^3 x \cos x\, dx$.

7. $\displaystyle\int \sin^3 x\, dx$. HINT: $\sin^3 x = (1 - \cos^2 x) \sin x$.

8. $\int (x^3 + \sin x)dx.$

9. $\int (2 + \cos^2 x) \sin x \cos x \, dx.$

10. $\int (1 + \sin^2 x)^2 \sin 2x \, dx.$

11. $\int \sec^2 x \tan x \, dx.$

12. $\int (\tan x + \cot x)^2 dx.$

13. $\int \frac{\sin 2x \, dx}{4 - \cos^2 x}.$

14. $\int \frac{\sin x \, dx}{4 - \cos^2 x}.$

15. $\int \frac{(\cos x - \sin x)dx}{\cos x + \sin x}.$

16. $\int (4 + 2^x)2^x dx.$

17. $\int e^{\tan x} \sec^2 x \, dx.$

18. $\int (e^x + e^{-x})^2 dx.$

19. $\int \frac{2x^2 + 4x + 3}{x + 2} dx.$

HINT: Perform the division.

20. $\int \frac{2x + 11}{x + 4} dx.$

21. $\int \frac{x^3 + 4x^2 + 4x + 3}{x + 1} dx.$

22. $\int \frac{x^3 + 2x^2 + 4x + 9}{x^2 + 4} dx.$

23. $\int \frac{(x + 4)dx}{x^2 + 4x + 8}.$

24. $\int \frac{(x + 2)dx}{(x^2 + 4x + 8)^3}.$

25. $\int \frac{(6 + \log^2 x)}{x} \log x \, dx.$

26. $\int \frac{dx}{9x^2 + 1}.$

27. $\int \frac{(2x + 4)dx}{\sqrt{1 - 4x^2}}.$

28. $\int \frac{\cos x}{e^{\sin x}} dx.$

29. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx.$

30. $\int \frac{2 \, dx}{x^{\frac{1}{2}}(3 + x^{\frac{1}{2}})}$

31. $\int \frac{4 \sec^2 x \, dx}{\tan^2 x + 4}.$

32. $\int \frac{dt}{2t(\log^2 t + 1)}.$

33. $\int \frac{e^{\sin x} \cos x \, dx}{\sqrt{4 - e^{2 \sin x}}}.$

34. $\int \frac{dx}{x\sqrt{4 - \log^2 x}}.$

35. $\int \frac{e^{2x}}{e^x + 4} dx.$

36. $\int \frac{\sqrt{\log x - 7}}{x} dx.$

37. $\int \sqrt{\frac{3}{x}}(1 - \sqrt{x}) \, dx.$

38. $\int (2\pi)^{5s} ds.$

39. $\int \frac{\tan 2x}{\cot 2x} dx.$

40. $\int \frac{dx}{1 + \cos x}.$

41. $\int \frac{(x - 3)dx}{4x^2 - 4x + 7}.$

42. $\int \frac{(3x - 1)dx}{\sqrt{x^2 + x + 1}}.$

43. $\int \frac{(x + 4)dx}{\sqrt{12x - 4x^2}}.$

44. $\int \frac{(6x - 1)dx}{\sqrt{4x^2 - 4x + 10}}.$

CHAPTER XV

APPLICATIONS OF THE INDEFINITE INTEGRAL

78. Introduction.—By adding the constant of integration we have been writing down the *most general* function whose derivative is a given function. In many applications to geometry, physics, chemistry etc., some *particular* integral

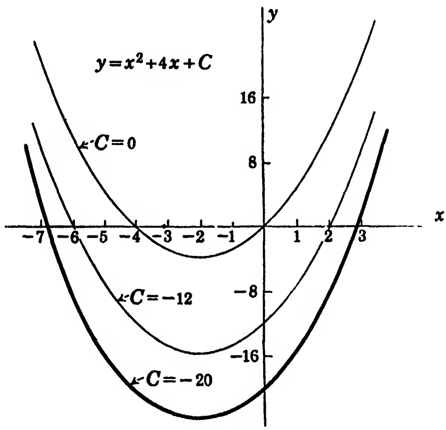

FIG. 68.

is required. From the available information one determines what value must be assinged to C in order to satisfy the conditions of the particular problem.

79. Applications to geometry.—Suppose that we wish to determine the equation of the curve through (3, 1) for which at any point

$$\frac{dy}{dx} = 2x + 4.$$

Integrating, we find at first that

$$y = x^2 + 4x + C.$$

This is the equation of a family of parabolas, one of which corresponds to each particular value of C, as indicated in Fig. 68. Any one of these parabolas satisfies the condition imposed upon the derivative. For the particular curve required we know that when $x = 3$, $y = 1$. Substituting and solving for C, we have

$$1 = 3^2 + 4(3) + C, \qquad \text{or} \qquad C = -20.$$

The equation of the required curve is then

$$y = x^2 + 4x - 20.$$

Somewhat more generally we may ask for the family of curves for which, at every point, the slope of the tangent is any given function of the coordinates; *i.e.*, for which

$$\frac{dy}{dx} = \varphi(x, y).$$

The solution of such a problem is in general difficult or impossible. It can be accomplished in certain simple cases by *separating the variables; i.e.*, by putting all of the y terms together with dy on one side, and all of the x terms together with dx on the other. When this can be done, each side may be integrated separately. The constant of integration may be added on either side and in any form; it is sometimes more convenient to add it in some such form as $\log C$ or C^2.

Example

What curve through (0, 2) has at every point the slope of its tangent equal to twice the product of the coordinates of that point?

Solution

We are given that at any point (x, y),

$$\frac{dy}{dx} = 2xy.$$

Separating the variables we have

$$\frac{dy}{y} = 2x\,dx.$$

Integrating both sides "simultaneously" we find

$$\log y = x^2 + C.$$

Since $y = 2$ when $x = 0$, $C = \log 2$. The equation of the required curve is then

$$\log y = x^2 + \log 2$$

or

$$\log \frac{y}{2} = x^2$$

or

$$y = 2e^{x^2}.$$

PROBLEMS

1. At every point of a certain curve $dy/dx = x^2 + 4$; the curve goes through $A(3, 9)$. What is its equation?

2. Prove that a curve, having its slope at every point proportional to the abscissa of the point, is a parabola.

3. For what curve through (0, 1) is the slope of the tangent equal to the ordinate?

4. For what curve through (1, −1) is the slope of the tangent equal to three times the square of the ordinate?

5. What curve through (3, 4) has the property that, at every point, the slope of its tangent is $-x/y$?

6. Find the equation of the curve passing through (3, 0) and having at every point $dy/dx = 3x^2 - 6x - 4$. Sketch the curve.

7. What curve through (1, 4) cuts all of the parabolas $y = x^2 + k$ at right angles? Sketch the curves. HINT: For the given curves $dy/dx = 2x$; hence, for the required curve $dy/dx = -1/2x$.

8. Find the equation of a curve through (4, 2) which intersects all of the hyperbolas $xy = k$ at right angles. See hint, Prob. 7.

9. Determine a family of curves such that each one cuts every member of the family of circles $x^2 + y^2 - 2x - 2y = C$ at right angles. Sketch both sets of curves. See hint, Prob. 7.

10. What family of curves has the property that each one cuts all of the hyperbolas $x^2 - y^2 = a^2$ at right angles? Sketch the curves.

11. What curve through (2, 2) has at every point

$$\frac{dy}{dx} = \frac{xy - x}{y}?$$

12. At every point of a certain curve $d^2y/dx^2 = 4x$; the curve passes through (1, 4) and the tangent at this point is inclined at 45°. What is the equation of this curve?

80. Physical problems. The compound interest law.—In many physical problems, the law governing the time rate of change of a function Q is known. The value of Q for a particular value (or values) of t is observed. With this information one can often, by integration, derive the relation between Q and t and thus compute the value of Q corresponding to any value of t.

Of particular interest is the case in which the rate at which Q is changing at any instant is proportional to the value at Q at that instant.

Example

The rate at which a certain substance in a solution is decomposing at any instant is known to be proportional to the amount of it present in the solution at that instant. Initially there are 27 g. and 3 hr. later it is found that 8 g. are left. How much will be left after another hour?

Solution

If Q is the amount present at any time t, we are given that

$$\frac{dQ}{dt} = kQ; \qquad \text{also} \qquad \begin{cases} Q = 27 \text{ when } t = 0 \\ Q = 8 \text{ when } t = 3. \end{cases}$$

Separating the variables and integrating we have

$$\begin{aligned} \frac{dQ}{Q} &= k\,dt \\ \log Q &= kt + \log C \\ Q &= Ce^{kt}. \end{aligned}$$

Since $Q = 27$ when $t = 0$, we find $C = 27$. Hence,

$$Q = 27e^{kt}.$$

The value of k may be found from the condition that $Q = 8$ when $t = 3$. Thus,

$$\begin{aligned} 8 &= 27e^{3k} \\ e^{3k} &= \tfrac{8}{27} \\ e^k &= \tfrac{2}{3}. \end{aligned}$$

Since $e^{kt} = (e^k)^t$ and $e^k = \frac{2}{3}$, we have finally

$$\mathbf{Q = 27\ (\tfrac{2}{3})^t.}$$

At the end of another hour $t = 4$ and

$$Q = 27(\tfrac{2}{3})^4 = 5\tfrac{1}{3} \text{ g}.$$

The graphical picture showing how Q decreases with increasing t is presented in Fig. 69. It has the familiar form of the exponential curve $y = a^{-x}$ where $a > 1$. The relation could of course be written in the form $Q = 27(\frac{3}{2})^{-t}$.

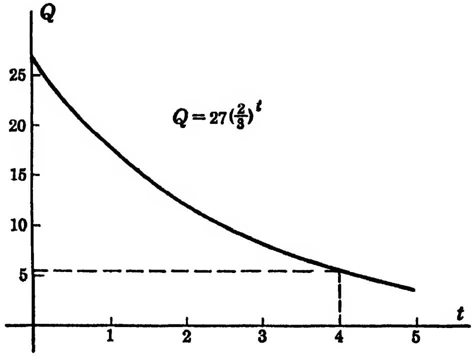

FIG. 69.

When a quantity Q increases at a rate which is at every instant proportional to the value of Q, it is said to increase according to the *compound interest law*. This is the way in which an amount of money would increase if the interest were added *continuously* instead of at the end of each year or other interest period.

Example

A sum of \$100 is invested at an interest rate of 6 per cent per annum compound continuously. What is the amount at the end of 10 years? Compare the situation with the cases in which it is compounded annually and semiannually.

Solution

Let A be the amount at any time t. Then $A = 100$ when $t = 0$ and we must find A when $t = 10$. At any instant, A is increasing at a rate of $0.06A$ dollars per year; *i.e.*,

$$\frac{dA}{dt} = 0.06A.$$

Separating the variables and integrating we have

$$\frac{dA}{A} = 0.06\,dt$$
$$\log A = 0.06\,t + C.$$

Since $A = 100$ when $t = 0$, $C = \log 100$; hence,

$$\log A = 0.06t + \log 100$$

$$\log \frac{A}{100} = 0.06t$$

$$A = 100e^{0.06t}.$$

This relation gives the amount A at any time t; taking $t = 10$ we have

$$A]_{t=10} = 100e^{0.6} = \$182.21.$$

Using tables it is found that the amount would be \$179.08 if compounded annually, and \$180.61 if compounded semiannually. If the amount were computed on the assumption of compounding monthly, daily, hourly, etc., the results would approach closer and closer to that obtained above as the length of the interest period approached zero.

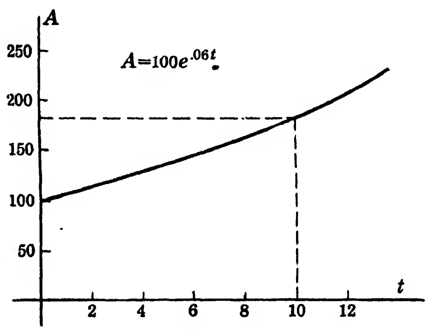

FIG. 70.

The graph of the relation between A and t is the exponential curve shown in Fig. 70.

PROBLEMS

1. In chemical reaction a substance A is being transformed into another substance B at a rate which is proportional to the amount of A remaining untransformed. Starting with 8 g. of A it is found that 4 g. remain after one-half hour. Express the amount remaining at any time as a function of t. How much remains after 2 hours?

2. Bacteria multiply at a rate proportional to the number present. If the original number N_1 doubles in an hour, in how many hours will there be 10 N_1?

3. A sum of \$100 is placed at interest at 4 per cent per annum compounded continuously. In how many years will the amount be doubled?

4. A sum of \$300 is invested at 5 per cent compounded continuously for 12 years. What is the final amount? Compare the result with the case in which it is compounded semiannually.

5. Suppose that a man could save money continuously at a rate of $1800 per year and could keep it invested at 6 per cent. In how many years would he have $20,000? HINT: Let A be the amount at any time; then A is increasing at a rate of $dA/dt = 1{,}800 + 0.06A$ dollars per year. Why?

6. A cylindrical tank containing 100 gal. of water stands on end. There is a small leak at the bottom. The rate at which water escapes is proportional to the square root of the depth (and hence proportional to the square root of the volume present, since the cross section is constant). If 10 gal. leaks out the first day, how much will leak out the next day? The fifth day?

7. A tank contains 1,000 gal. of brine in which there are 600 lb. of dissolved salt. Pure water is run into the tank at 12 gal. per minute while the mixture is withdrawn at the same rate. The solution is kept uniform by stirring. In how many minutes will the concentration be reduced to $\frac{1}{10}$ lb. of salt per gallon? HINT: Let P be the number of lb. of salt present at any time; then the concentration is $P/1{,}000$ lb. per gallon and in withdrawing 12 gal. per minute one withdraws $12P/1{,}000$ lb. of salt per minute; *i.e.*, $dP/dt = -12P/1{,}000$.

8. Solve Prob. 7 if pure water is run in at 16 gal. per minute, the mixture being withdrawn at 12 gal. per minute.

9. In Prob. 7 how many minutes would it take to reduce the amount of salt to 100 lb. if the mixture is withdrawn at 12 gal. per minute while water is run in at 16 gal. per minute?

10. A tank contains 600 gal. of brine in which 200 lb. of salt are dissolved. Water is run into the tank at 12 gal. per minute and the mixture is withdrawn at 8 gal. per minute, the solution being kept uniform. In how many minutes will the amount of salt be reduced to 40 lb.?

11. A tank contains 25 gal. of water and 25 gal. of alcohol mixed together. Alcohol is run in at 5 gal. per minute while the mixture is withdrawn at 3 gal. per minute. The mixture is kept uniform by stirring. In how many minutes will the contents be 75 per cent alcohol? HINT: Let W be the number of gallons of water present at any time.

12. Assume that the rate at which a body cools in air is proportional to the difference between its temperature and that of the surrounding atmosphere. A body originally at 80° cools down to 60° in 20 min., the air temperature being 40°. What will be its temperature after another 40 min.?

13. The temperature of a body drops from 100 to 80° in 3 hr. when surrounded by air having a temperature of 20°. Assuming the law of cooling stated in the preceding problem, in how long will the temperature of the body be reduced to 30°?

81. Rectilinear motion. The projectile.—If a particle whose weight is W lb. is acted on by a resultant force of

F lb., it is known that an acceleration is produced in the direction of the applied force. The relation between the magnitudes of the applied force, the weight of the particle, and the acceleration produced (in feet per second per second) is experimentally found to be

$$F = \frac{W}{g}a.$$

In this equation g is a constant whose value is 32.2. The quantity W/g is called the *mass* of the particle.

In many simple problems involving rectilinear motion, both the force F and the weight W are known. The acceleration can then be obtained from the above relation. And, since $a = dv/dt = d^2s/dt^2$, one can, by two integrations, obtain expressions for the velocity and displacement of the particle in terms of the time.

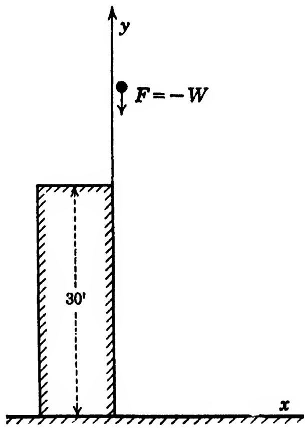

FIG. 71.

Example

From a point 30 ft. above the ground, a small lead ball is thrown vertically upward with an initial velocity of 60 ft. per second. Find its velocity and distance *from the ground* at the end of t sec.

Solution (Fig. 71)

The only force acting on the ball while in the air (neglecting air resistance) is its own weight which acts vertically *downward*. If we take the upward direction as positive we have

$$-W = \frac{W}{g}a;$$

$$a = -g$$

or

$$\frac{dv}{dt} = -g.$$

Integrating we obtain

$$v = -gt + C_1.$$

Since $v = +60$ when $t = 0$, $C_1 = 60$; hence, at any time,

$$v = 60 - gt.$$

Replacing v by dy/dt and integrating again we have

$$y = 60t - \tfrac{1}{2}gt^2 + C_2.$$

Since $y = +30$ when $t = 0$, $C_2 = 30$; hence, at any time,

$$y = 30 + 60t - \tfrac{1}{2}gt^2.$$

In the more general problem, the projectile is fired with initial velocity v_0 at an angle α with the horizontal as indicated in Fig. 72. The motion in this case is not rectilinear. However, the only force acting on the projectile while it is in the air (neglecting air resistance) is again

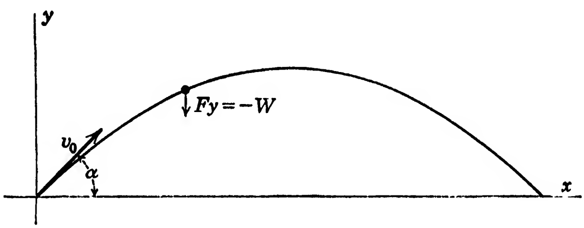

FIG. 72.

its own weight which of course acts vertically downward. Choosing axes as indicated in the figure, we may write our fundamental relation in the form

$$F_x = \frac{W}{g}a_x, \qquad F_y = \frac{W}{g}a_y.$$

Substituting 0 for F_x and $-W$ for F_y, and solving for the components of the acceleration we have

$$a_x = 0, \qquad a_y = -g.$$

Integrating and evaluating the constants we find that, at any time t,

$$v_x = v_0 \cos \alpha, \qquad v_y = v_0 \sin \alpha - gt.$$

Integrating again and evaluating the constants we have

$$x = (v_0 \cos \alpha)t, \qquad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2.$$

The details are left to the student as an exercise. The last two equations may of course be regarded as parametric equations of the path followed by the projectile.

PROBLEMS

1. From the top of a building 50 ft. high, a stone is thrown vertically upward with initial velocity 80 ft. per second. Express its velocity and distance above the ground at any time t as functions of t. Find the greatest height reached by the projectile.

2. From a point 800 ft. above the ground a small heavy particle is thrown vertically downward with initial velocity 60 ft. per second. Derive expressions for its velocity and distance from the ground at any time t. With what velocity does it strike the ground?

3. Derive expressions for the velocity and position at any time t of a body dropped from a height of h ft. above the earth. Show that its velocity when it strikes the earth will be $\sqrt{2gh}$ if the air resistance is neglected.

4. An airplane at an elevation of 6,000 ft. is moving horizontally at 100 ft. per second when a small lead ball is dropped. Where and with what velocity will it strike the ground? Illustrate with a sketch.

5. Derive the equations of motion for a projectile which is fired with initial velocity v_0 at an angle α with the horizontal. See Fig. 72.

6. A particle of weight W is projected vertically downward with initial velocity of 20 ft. per second through a resisting medium. If the resistance is equal to kv^2 where $k = W/100$, derive an expression for its velocity at any time t. Discuss the way in which v varies and show that as time goes on, it approaches 10 ft. per second downward. HINT: The *resultant* force acting on the projectile is $W - kv^2$.

7. A body is projected horizontally with initial velocity v_0 through a medium in which the resisting force is proportional to the velocity of the body. Show that the velocity will decrease according to the law $v_x = v_0e^{-kt}$ and that the horizontal distance moved in t seconds is $x = \frac{v_0}{k}(1 - e^{-kt})$.

8. Suppose that an automobile of weight W starts from rest and is supplied by the motor with a *constant* driving force F. If the resistance to motion is proportional to the velocity show that the car will gain speed according to the law

$$v = \frac{F}{k}\left(1 - e^{-\frac{gkt}{W}}\right).$$

9. A particle of weight W falls from rest in a medium in which the resistance to motion is equal to kv. Derive an expression for its velocity at any time. Show that as time goes on its velocity approaches W/k.

10. A bullet is fired horizontally from a point A with velocity v_0. The air resistance may be assumed to be horizontal and equal to $-kv_x^2$.

Derive expressions for v_x and v_y in terms of t. Find also the horizontal and vertical distances x and y of the bullet from A at any time t.

11. A block weighing 200 lb. is propelled along a rough horizontal surface, starting from rest, by a horizontal force of 40 lb. There is a frictional resistance of 30 lb. between the block and the surface. The resistance of the medium is equal to $2v$ where v is the velocity of the block. Derive expressions for the velocity and displacement of the block in terms of t. HINT: $F = 40 - 30 - 2v$.

12. A man and parachute together weigh 144 lb. The resistance of the air to the parachute varies jointly as the projected area on a plane perpendicular to the direction of motion and the square of the velocity. This resisting force is 1 lb. per square foot of area when $v = 20$ ft. per second. If $v = 60$ ft. per second when the parachute is opened and the area is 100 sq. ft., show that at any subsequent time v is given by

$$\frac{v - 24}{v + 24} = \frac{3}{7}e^{-\frac{gt}{12}}.$$

Show that v approaches 24 ft. per second as a limit. HINT: Taking positive direction downward, show first that $F = 144 - \frac{v^2}{4}$; hence

$$a = -\frac{g}{24^2}(v^2 - 24^2).$$

CHAPTER XVI

TRIGONOMETRIC INTEGRALS

82. Introduction.—Many problems of the integral calculus lead either directly or indirectly to integrals involving powers of the trigonometric functions. Using the fundamental identities and the double- and half-angle formulas one can often transform such integrals into forms in which the standard integration formulas can be applied. Thus, none of the formulas applies directly to $\int \cos^3 x\,dx$; however

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x \cos x\,dx\\ &= \int (1 - \sin^2 x) \cos x\,dx\\ &= \int \cos x\,dx - \int \sin^2 x \cos x\,dx\\ &= \sin x - \tfrac{1}{3}\sin^3 x + C.\end{aligned}$$

In this chapter we shall consider only the types of trigonometric integrals which arise often in the later applications and for which simple rules for integration can be given.

83. $\int \sin^m x \cos^n x\,dx$.—Simple rules can be given for handling this integral if one of the exponents is a positive *odd* integer or if both exponents are positive *even* integers. Since the treatments are entirely different they will be given separately.

CASE I. *One of the exponents a positive odd integer.*—This case can always be handled by the procedure just used on $\int \cos^3 x\,dx$. Suppose for example that m is odd. One can then take out $\sin x\,dx$ as dv leaving an *even* exponent for $\sin x$; then, using the relation

$$\sin^2 x = 1 - \cos^2 x$$

he can obtain a series of terms of the form

$$\int \cos^q x \sin x\,dx$$

which can be integrated by the formula for $\int v^n dv$.

Example

$$\int \sin^3 x \cos^4 x \, dx$$

Solution

Since the exponent of $\sin x$ is a positive odd integer we proceed as follows:

$$\begin{aligned}
\int \sin^3 x \cos^4 x \, dx &= \int \cos^4 x \sin^2 x \sin x \, dx \\
&= \int \cos^4 x (1 - \cos^2 x) \sin x \, dx \\
&= \int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx \\
&= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.
\end{aligned}$$

The student should notice carefully that this procedure will always apply if one of the exponents is a *positive odd integer* no matter what the other exponent may be—any positive or negative integer or fraction, or zero.

CASE II. *Both exponents positive even integers.*—The integration can be accomplished in this case by changing over to multiple angles. For this purpose the following formulas, which the student can easily derive, are used:

$$\begin{aligned}
\sin^2 x &= \tfrac{1}{2}(1 - \cos 2x). \\
\cos^2 x &= \tfrac{1}{2}(1 + \cos 2x). \\
\sin x \cos x &= \tfrac{1}{2} \sin 2x.
\end{aligned}$$

Example 1

$$\int \cos^2 x \, dx.$$

Solution

$$\begin{aligned}
\int \cos^2 x \, dx &= \int \tfrac{1}{2}(1 + \cos 2x) dx \\
&= \tfrac{1}{2}\int dx + \tfrac{1}{2}\int \cos 2x \, dx \\
&= \tfrac{1}{2}x + \tfrac{1}{4} \sin 2x + C.
\end{aligned}$$

Example 2

$$\int \sin^2 x \cos^2 x \, dx.$$

Solution

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \tfrac{1}{4}\int \sin^2 2x \, dx \\
&= \tfrac{1}{4}\int \tfrac{1}{2}(1 - \cos 4x) dx \\
&= \tfrac{1}{8}x - \tfrac{1}{32} \sin 4x + C.
\end{aligned}$$

PROBLEMS

1. $\int \sin^3 x\,dx.$

2. $\int \cos^3 \frac{1}{2}x\,dx.$

3. $\int \sin^5 x\,dx.$

4. $\int \cos^5 2x\,dx.$

5. $\int \sin^7 x\,dx.$

6. $\int \cos^7 x\,dx.$

7. $\int \sin x \cos^5 x\,dx.$

8. $\int \sin^3 2\theta \cos^3 2\theta\,d\theta.$

9. $\int \sin^3 x \cos^2 x\,dx.$

10. $\int \sin^2 x \cos^3 x\,dx.$

11. $\int \sin^5 x \cos^5 x\,dx.$

12. $\int \sqrt{\sin x}\cos^3 x\,dx.$

13. $\int \tan^3 x\,dx.$

14. $\int \sin^4 x \cos^3 x\,dx.$

15. $\int \sin x \sec^2 x\,dx.$

16. $\int \tan\theta \sec^3\theta\,d\theta.$

17. $\int \frac{\sec^4 x\,dx}{\csc^3 x}.$

18. $\int \frac{\sin^3 x\,dx}{\sqrt{\cos x}}.$

19. $\int \sin^2 x\,dx.$

20. $\int \cos^2 \frac{1}{2}\theta\,d\theta.$

21. $\int \sin^4 x\,dx.$

22. $\int \cos^4 x\,dx.$

23. $\int \sin^6 x\,dx.$

24. $\int \cos^6 \frac{1}{2}x\,dx.$

25. $\int \sin^2 \frac{1}{2}\theta \cos^2 \frac{1}{2}\theta\,d\theta.$

26. $\int \sin^2 x \cos^4 x\,dx.$

27. $\int \sin^4 x \cos^4 x\,dx.$

28. $\int \sin^4 x \cos^2 x\,dx.$

29. $\int (\sin x + \cos x)^2 dx.$

30. $\int (1 + \sin 2x)^2 dx.$

31. $\int (\cos 2x + \cos x)^2 dx.$

32. $\int (\sqrt{\sin x} + 2\cos x)^2 dx.$

33. $\int \sin x \sin 2x\,dx.$

34. $\int \tan^2 x \cos^3 x \sin x\,dx.$

35. $\int \sin x \cos x \cos^4 2x\,dx.$

36. $\int \sin^2 \frac{1}{2}\theta \cos^2\theta\,d\theta.$

37. $\int \cos^4\theta \tan^2\theta\,d\theta.$

38. From the relation $\cos 2\theta = \cos^2\theta - \sin^2\theta$, show that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$.

84. $\int \sec^n x\,dx$ or $\int \csc^n x\,dx$.—If n is any *positive even integer* the integration can be performed as indicated in the following example:

Example

$$\int \sec^6 x\,dx$$

Solution

We may take out $\sec^2 x\,dx$ as dv and transform the remainder into a polynomial in $\tan x$; thus,

$$\begin{aligned}
\int \sec^6 x\,dx &= \int \sec^4 x \sec^2 x\,dx \\
&= \int (1 + \tan^2 x)^2 \sec^2 x\,dx \\
&= \int (1 + 2\tan^2 x + \tan^4 x) \sec^2 x\,dx \\
&= \int \sec^2 x\,dx + 2\int \tan^2 x \sec^2 x\,dx + \int \tan^4 x \sec^2 x\,dx \\
&= \tan x + \tfrac{2}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x + C.
\end{aligned}$$

If n is a positive *odd* integer, a method called *integration by parts* may be employed. This method will be discussed in the next chapter.

85. $\int \tan^n x\,dx$ or $\int \cot^n x\,dx$.—If n is any *positive integer*, these types can be reduced to forms which are easily integrated by the use of the relations:

$$\tan^2 x = \sec^2 x - 1.$$
$$\cot^2 x = \csc^2 x - 1.$$

Example

$$\begin{aligned}
\int \tan^4 x\,dx &= \int \tan^2 x(\sec^2 x - 1)dx \\
&= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx \\
&= \tfrac{1}{3}\tan^3 x - \int(\sec^2 x - 1)dx \\
&= \tfrac{1}{3}\tan^3 x - \tan x + x + C.
\end{aligned}$$

86. $\int \tan^m x \sec^n x\,dx$ or $\int \cot^m x \csc^n x\,dx$.—We shall consider here only two cases.

CASE I. *Exponent of* $\sec x$ *a positive even integer.*—In this case one may take out $\sec^2 x\,dx$ as dv and transform the remainder into a polynomial in $\tan x$.

Example

$$\int \tan^3 x \sec^4 x\,dx.$$

Solution

Since the exponent of $\sec x$ is a positive *even* integer we may proceed as follows:

$$\begin{aligned}\int \tan^3 x \sec^4 x\, dx &= \int \tan^3 x \sec^2 x \sec^2 x\, dx\\ &= \int \tan^3 x\,(1+\tan^2 x)\sec^2 x\, dx\\ &= \int \tan^3 x \sec^2 x\, dx + \int \tan^5 x \sec^2 x\, dx\\ &= \tfrac{1}{4}\tan^4 x + \tfrac{1}{6}\tan^6 x + C.\end{aligned}$$

CASE II. *Exponent of tan x a positive odd integer.*—In this case one can take out $\sec x \tan x\, dx$ as dv and transform the remainder into a polynomial in $\sec x$.

Example

$$\int \sec^3 x \tan^3 x\, dx$$

Solution

Since the exponent of $\tan x$ is a positive *odd* integer we proceed as follows:

$$\begin{aligned}\int \sec^3 x \tan^3 x\, dx &= \int \sec^2 x \tan^2 x \sec x \tan x\, dx\\ &= \int \sec^2 x\,(\sec^2 x - 1)\sec x \tan x\, dx\\ &= \int \sec^4 x \sec x \tan x\, dx - \int \sec^2 x \sec x \tan x\, dx\\ &= \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C.\end{aligned}$$

It is of course possible that a given integral could be handled by either of these procedures. Thus, $\int \sec^4 x \tan^3 x\, dx$ could be found by either method since the exponent of $\sec x$ is even and that of $\tan x$ is odd. On the other hand, *neither* of the procedures could be used on $\int \sec^3 x \tan^2 x\, dx$.

PROBLEMS

1. $\displaystyle\int \tan^2 x\, dx.$
2. $\displaystyle\int \tan^3 x\, dx.$
3. $\displaystyle\int \cot^2 \tfrac{1}{2}\theta\, d\theta.$
4. $\displaystyle\int \cot^3 2\theta\, d\theta.$
5. $\displaystyle\int \tan^4 \theta\, d\theta.$
6. $\displaystyle\int \cot^4 \tfrac{1}{2}x\, dx.$
7. $\displaystyle\int \tan^5 2x\, dx.$
8. $\displaystyle\int \sec^2 \tfrac{1}{2}x\, dx.$
9. $\displaystyle\int \sec^4 \tfrac{1}{2}x\, dx.$
10. $\displaystyle\int \sec^6 4x\, dx.$
11. $\displaystyle\int \frac{dx}{\sin^2 x}.$
12. $\displaystyle\int \frac{dx}{\cos^4 x}.$
13. $\displaystyle\int \sec^2 x \tan^3 x\, dx.$
14. $\displaystyle\int \csc^2 x \cot^3 x\, dx.$

15. $\int \sec^6 x \tan x \, dx.$

16. $\int \sec^4 2x \tan^4 2x \, dx.$

17. $\int \sec^2 \theta \tan^4 \theta \, d\theta.$

18. $\int \sec^6 2x \tan^2 2x \, dx.$

19. $\int \sec x \tan^3 x \, dx.$

20. $\int \csc^3 x \cot^3 x \, dx.$

21. $\int \sec^5 x \tan^3 x \, dx.$

22. $\int \tan^5 x \sec x \, dx.$

23. $\int \sec^3 x \tan^5 x \, dx.$

24. $\int \tan^5 x \sec^4 x \, dx.$

25. $\int \sin 2x \sec^6 x \, dx.$

26. $\int \frac{\sin^2 x}{\cos^4 x} dx.$

27. $\int \sin 2x \cos x \tan^2 x \, dx.$

28. $\int \sqrt{\tan x} \sec^4 x \, dx.$

29. $\int (\tan x + \cot x)^2 dx.$

30. $\int (2 \sec x + 3 \tan x)^2 dx.$

CHAPTER XVII

METHODS OF INTEGRATION

87. Introduction.—A study of integration is largely a study of methods of transforming various types of integrands into forms in which the fundamental integration formulas can be applied. One of the most useful devices for this purpose is that of substituting a new variable. Consider as an example

$$\int \sqrt{4 - x^2}\, dx.$$

The integration cannot be performed directly by any of the standard formulas. Let us, then, transform the integrand into a rational trigonometric function by letting

$$x = 2 \sin \theta.$$

Since if $x = 2 \sin \theta$, $dx = 2 \cos \theta\, d\theta$, we have, upon substituting,

$$\begin{aligned} \int \sqrt{4 - x^2}\, dx &= \int \sqrt{4 - 4 \sin^2 \theta}\, 2 \cos \theta\, d\theta \\ &= 4\int \cos^2 \theta\, d\theta \\ &= 4\int \tfrac{1}{2}(1 + \cos 2\theta) d\theta \\ &= 2\theta + \sin 2\theta + C. \end{aligned}$$

Having thus performed the integration in terms of θ, we can change the result back into terms of x as follows: Since $x = 2 \sin \theta$,

$$\sin \theta = \frac{x}{2}.$$

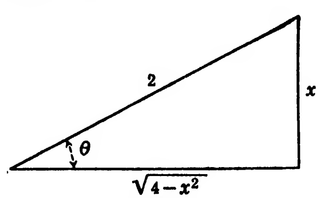

FIG. 73.

The relation between θ and x is therefore represented by the triangle shown in Fig. 73. From it,

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$= 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4 - x^2}}{2}.$$

Finally then,

$$\int \sqrt{4 - x^2}\, dx = 2 \arcsin \frac{x}{2} + \frac{x\sqrt{4 - x^2}}{2} + C.$$

88. Trigonometric substitutions.—If the integrand contains one of the radicals,

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2},$$

it can be transformed into a rational trigonometric function as follows:

If $\sqrt{a^2 - x^2}$ occurs, let $x = a \sin \theta$.
If $\sqrt{a^2 + x^2}$ occurs, let $x = a \tan \theta$.
If $\sqrt{x^2 - a^2}$ occurs, let $x = a \sec \theta$.

If the resulting trigonometric integral comes under one of the cases discussed in Chap. XVI, the integration can be performed and the result changed back into terms of x as illustrated in the preceding article.

The student is warned against the common error of merely substituting $d\theta$ for dx. One must actually substitute for dx its value in terms of θ and $d\theta$. Thus, if one lets

$$x = a \tan \theta$$

then *differentiating*,

$$\frac{dx}{d\theta} = a \sec^2 \theta$$

and

$$dx = a \sec^2 \theta\, d\theta.$$

PROBLEMS

1. Derive a formula for $\int \frac{dv}{\sqrt{v^2 - a^2}}$. *Formula* (16)

2. Derive a formula for $\int \sqrt{a^2 - v^2}\, dv$.

3. $\displaystyle\int \frac{x^3dx}{\sqrt{4 + x^2}}$.

4. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2 + x^2}}$.

5. $\displaystyle\int \frac{dx}{x^4\sqrt{5 + x^2}}$.

6. $\displaystyle\int \frac{\sqrt{x^2 + 9}}{x^4}dx$.

7. $\displaystyle\int x^3\sqrt{16 + x^2}\, dx$.

8. $\displaystyle\int x^2\sqrt{3 - x^2}\, dx$.

9. $\displaystyle\int x^3\sqrt{9 - x^2}\, dx$.

10. $\displaystyle\int \frac{\sqrt{4 - 9x^2}}{x^2}dx$.

11. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$.

12. $\displaystyle\int \frac{x^3dx}{\sqrt{a^2 - x^2}}$.

13. $\displaystyle\int \frac{dt}{t\sqrt{4 - t^2}}$.

14. $\displaystyle\int \frac{\sqrt{x^2 - 4}}{x}dx$.

15. $\displaystyle\int z^3\sqrt{z^2 - 6}\, dz$.

16. $\displaystyle\int \frac{\sqrt{y^2 - 16}}{y^4}dy$.

17. $\displaystyle\int \frac{x^3dx}{\sqrt{x^2 - 4}}$.

18. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}}$.

19. $\displaystyle\int \frac{dt}{t^2\sqrt{t^2 - 16}}$.

20. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2 - 8}}$.

21. $\displaystyle\int x(4 - x^2)^{\frac{3}{2}}dx$.

22. $\displaystyle\int x\sqrt{(x^2 - 16)^3}\, dx$.

23. $\displaystyle\int x^3(a^2 - x^2)^{\frac{3}{2}}dx$.

24. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$.

25. $\displaystyle\int x\sqrt{3 - 2x - x^2}\, dx$.

26. $\displaystyle\int \frac{x^2dx}{\sqrt{(a^2 - x^2)^3}}$.

27. $\displaystyle\int \frac{x^2dx}{\sqrt{(x^2 + a^2)^3}}$.

28. $\displaystyle\int \frac{x^2dx}{\sqrt{(a^2 - x^2)^5}}$.

89. Algebraic substitutions.—If the integrand contains one term of the form

$$\sqrt[n]{ax + b}$$

it can be rationalized by the substitution

$$ax + b = z^n.$$

Example

Evaluate $\displaystyle\int \frac{x^2dx}{\sqrt{3 + 4x}}$.

Solution

Let $3 + 4x = z^2$, then $dx = z\,dz/2$, and we have

$$\int \frac{x^2dx}{\sqrt{3 + 4x}} = \int \frac{\left(\frac{z^2 - 3}{4}\right)^2 \frac{z dz}{2}}{z}.$$

$$= \frac{1}{32}\int (z^4 - 6z^2 + 9)dz$$

$$= \frac{1}{32}\left[\frac{z^5}{5} - 2z^3 + 9z\right] + C$$

$$= \frac{1}{32}\left[\frac{(3 + 4x)^{\frac{5}{2}}}{5} - 2(3 + 4x)^{\frac{3}{2}} + 9(3 + 4x)^{\frac{1}{2}}\right] + C.$$

90. Substitutions in general.—The transformations just studied constitute some of the most useful ones; they do not by any means exhaust the possibilities. One is of course free at any time to make any change of variable that he pleases. If the integration can be performed in terms of the new variable, the result can easily be chânged back into terms of the original variable.

A considerable amount of ingenuity is required in the matter of choosing a substitution which will simplify a given integral. The following examples will give the student some idea of the many possibilities.

1. The substitution $e^x = z$ reduces

$$\int \frac{dx}{e^x - 1} \quad \text{to} \quad \int \frac{dz}{z^2 - z}.$$

2. The substitution $\tan \frac{1}{2}x = z$ reduces

$$\int \frac{dx}{\sin x - 4 \cos x - 4} \quad \text{to} \quad \int \frac{dz}{z - 4}.$$

3. The substitution $x = 1/z$ reduces

$$\int \frac{dx}{x^2\sqrt{a^2 + x^2}} \quad \text{to} \quad -\int \frac{z\,dz}{\sqrt{1 + a^2z^2}}.$$

PROBLEMS

1. $\int \frac{\sqrt{x}\,dx}{\sqrt{x} + 4}.$

2. $\int \frac{\sqrt{x} + 2}{\sqrt{x} - 1}dx.$

3. $\int \frac{\sqrt{1+x}}{x-3}dx.$

4. $\int \frac{dx}{\sqrt{x+4}+8}.$

5. $\int x\sqrt[3]{3x+4}\,dx.$

6. $\int \frac{dt}{\sqrt[3]{t}+\sqrt{t}}.$

7. $\int \frac{\sqrt{x}\,dx}{x+1}.$

8. $\int \frac{x^2dx}{\sqrt[3]{x+1}}.$

9. $\int x^5\sqrt{2x^3+4}\,dx.$

10. $\int x^3\sqrt{x^2+6}\,dx.$

11. Show that if one lets

$$\tan\frac{x}{2} = z, \qquad i.e., \qquad x = 2 \text{ arc } \tan z$$

then

$$dx = \frac{2\,dz}{1+z^2},$$

$$\sin x = \frac{2z}{1+z^2},$$

$$\cos x = \frac{1-z^2}{1+z^2}.$$

12. Use the method of Prob. 11 to evaluate

$$\int \frac{dx}{\sin x - 2\cos x - 2}.$$

13. $\int \frac{dt}{2-\sin t}.$

14. $\int \frac{\sqrt{x-4}}{x\sqrt{x}}dx.$

HINT: Let $x = 4\sec^2\theta$.

15. $\int \sqrt{4-\sqrt{x}}\,dx.$

16. $\int \frac{e^x(e^x-4)}{e^x+2}dx.$

17. $\int \frac{dx}{x\sqrt{2rx-x^2}}.$

HINT: Let $x = 2r\sin^2\theta$.

18. $\int \frac{dx}{x\sqrt{x^2-2rx}}.$

91. Integration by parts.—One of the most useful aids to integration is the process known as *integration by parts*. The necessary formula is obtained from that for the differential of a product. If u and v are differentiable functions of x then

$$d(uv) = u\,dv + v\,du.$$

Integrating both sides with respect to x, we may write

$$uv = \int u\,dv + \int v\,du.$$

Rearranging this result, we have the formula for integration by parts,

$$\int u\,dv = uv - \int v\,du.$$

In order to use this formula we must regard the given integrand as the product of a function u and the differential dv of another function v. No general rule for thus breaking the integrand into two parts can be given. One usually takes as much of it as he can readily integrate as dv and calls the remainder u.

Example

$$\int xe^x dx$$

Solution

We cannot perform the integration by previous methods. However we *can* integrate $e^x dx$. We may therefore regard the integrand as being of the form $u\,dv$ where,

$$u = x \qquad \text{and} \qquad dv = e^x dx.$$

Differentiating u to obtain du, and integrating dv to obtain v we have

$$du = dx \qquad \text{and} \qquad v = e^x.$$

Using the formula for integration by parts, we obtain

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C. \end{aligned}$$

A wrong choice of u and dv may lead to a more complicated integral. Suppose, for instance, that in the example just solved we had let

$$u = e^x \qquad dv = x\,dx;$$

then,

$$du = e^x dx \qquad v = \tfrac{1}{2}x^2.$$

Applying the formula we have

$$\int xe^x dx = \tfrac{1}{2}x^2e^x - \tfrac{1}{2}\int x^2e^x dx.$$

This relation is of course true, but it is readily seen that the second integral is more difficult to handle than the original one.

In some cases it may be necessary to apply the formula more than once. Thus, if one starts with

$$\int x^2 e^x dx,$$

the first application of the formula, letting $u = x^2$ and $dv = e^x dx$, leads to an integral of the form

$$\int x e^x dx.$$

This integral may in turn be evaluated by parts as in the previous example.

In certain cases it may be necessary to use the procedure illustrated by the following:

Example

$$\int e^x \sin x\, dx.$$

Solution

Let

$$u = e^x \qquad dv = \sin x\, dx;$$
$$du = e^x dx \qquad v = -\cos x.$$

Hence,

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx. \tag{1}$$

The integral obtained on the right-hand side is not simpler than the original one. However, we may apply integration by parts to it letting

$$u = e^x \qquad dv = \cos x\, dx;$$
$$du = e^x dx \qquad v = \sin x.$$

Substituting in (1) we obtain

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx.$$

Transposing, we have

$$2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + C'$$
$$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

PROBLEMS

1. $\int xe^{-x}dx.$
2. $\int xa^x dx.$
3. $\int x^2 e^{2x} dx.$
4. $\int x^2 a^x dx.$

5. $\int \log x\, dx.$

6. $\int x^2 \log x\, dx.$

7. $\int \log^2 x\, dx.$

8. $\int t \sin t \cos t\, dt.$

9. $\int x \cos 2x\, dx.$

10. $\int \theta^2 \sin 2\theta\, d\theta.$

11. $\int \operatorname{arc} \sin x\, dx.$

12. $\int \operatorname{arc} \sec \frac{1}{x} dx.$

13. $\int \operatorname{arc} \tan x\, dx.$

14. $\int e^x \cos x\, dx.$

15. $\int e^x \sin 2x\, dx.$

16. $\int e^{2x} \sin \tfrac{1}{2}x\, dx.$

17. $\int e^{\frac{x}{3}} \cos x\, dx.$

18. $\int x \sec^2 2x\, dx.$

19. $\int \frac{x\, dx}{\sin^2 3x}.$

20. $\int \sin x \sin 3x\, dx.$

21. $\int x \operatorname{arc} \tan x\, dx.$

22. $\int x^2 \operatorname{arc} \cot x\, dx.$

23. $\int \sec^3 x\, dx.$

24. $\int \frac{\tan^2 x\, dx}{\cos x}.$

25. $\int x \cos^2 \tfrac{1}{2}x\, dx.$

26. $\int x \sin^2 x\, dx.$

27. $\int \sin \sqrt{x}\, dx.$

HINT: First let $x = z^2$.

28. $\int (2x + e^x)^2 dx.$

29. $\int (x + \sin 2x)^2 dx.$

30. $\int \frac{(3x + 4 \sin 2x)}{\cos^2 x} dx.$

31. $\int \sec^5 x\, dx.$

32. $\int \sec^3 x \tan^2 x\, dx.$

33. Show that $\int x^m \log x\, dx = x^{m+1}\left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2}\right].$

34. Show that $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}.$

35. Show that $\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}.$

92. Integration of rational fractions.—Any rational fraction in which the numerator is *not* of lower degree than the denominator can be reduced, by performing the indicated division, to a simple polynomial plus a fraction in which the numerator *is* of lower degree than the denominator. Thus by actual division,

$$\frac{x^4 + 2x^3 - 5x^2 - 8x + 16}{x^3 - x^2 - 4x + 4} = x + 3 + \frac{2x^2 + 4}{x^3 - x^2 - 4x + 4}.$$

If one has such a fraction to integrate, his first step is to perform the division. Since the polynomial thus obtained is easily integrated, we need consider here only the integration of the remaining fraction in which the numerator is of *lower* degree than the denominator.

The integration of such a fraction is carried out by breaking it up into a sum of several simpler fractions called *partial fractions.* Thus, in the above example, it will be shown that

$$\frac{2x^2 + 4}{x^3 - x^2 - 4x + 4} \equiv \frac{1}{x + 2} + \frac{3}{x - 2} - \frac{2}{x - 1}.$$

The first step in thus breaking a given fraction up into partial fractions is to *factor the denominator* into its prime factors. The rest of the procedure depends upon the nature of the factors obtained. We shall consider only three cases.

CASE I. *Factors of denominator all linear and each occurring only once.*—Corresponding to each factor of the form $px + q$ assume a fraction of the form

$$\frac{A}{px + q}.$$

In the example just discussed, the factors of the denominator are $(x + 2)$, $(x - 2)$, and $(x - 1)$. Hence we assume that for proper values of A, B, and C, which are yet to be determined,

$$\frac{2x^2 + 4}{x^3 - x^2 - 4x + 4} \equiv \frac{A}{x + 2} + \frac{B}{x - 2} + \frac{C}{x - 1}$$
$$\equiv \frac{A(x - 2)(x - 1) + B(x + 2)(x - 1) + C(x + 2)(x - 2)}{(x + 2)(x - 2)(x - 1)}.$$

The denominators are identical; hence, the fractions will be identical if we determine A, B, and C, so that

$$\begin{aligned} 2x^2 + 4 &\equiv A(x - 2)(x - 1) + B(x + 2)(x - 1) \\ &\qquad + C(x + 2)(x - 2) \\ &\equiv (A + B + C)x^2 + (-3A + B)x \\ &\qquad + (2A - 2B - 4C). \end{aligned}$$

Two polynomials in x are equal for all values of x if, and only if, the coefficients of like powers of x are equal. Equating these coefficients we have

$$\begin{aligned} A + B + C &= 2 \\ -3A + B &= 0 \\ 2A - 2B - 4C &= 4. \end{aligned}$$

Solving these equations, we find

$$A = 1, \qquad B = 3, \qquad C = -2.$$

Hence,

$$\frac{2x^2 + 4}{x^3 - x^2 - 4x + 4} \equiv \frac{1}{x + 2} + \frac{3}{x - 2} - \frac{2}{x - 1}.$$

The result may of course be checked by combining the three fractions on the right into a single fraction.

A somewhat simpler method of finding the values of A, B, and C, is as follows: If

$$2x^2 + 4 \equiv A(x - 2)(x - 1) + B(x + 2)(x - 1) + C(x + 2)(x - 2)$$

for *all* values of x, then they must certainly be equal for any particular values of x which we may care to choose. Substituting for x the values 2, -2, and 1 successively, we find

$$\begin{aligned} 12 &= 4B, &\quad &\text{or} &\quad B &= 3. \\ 12 &= 12A, &\quad &\text{or} &\quad A &= 1. \\ 6 &= -3C, &\quad &\text{or} &\quad C &= -2. \end{aligned}$$

It is obvious that we may substitute *any* three values of x that we please and obtain three equations in the unknowns A, B, and C. The above values were chosen so that each equation would contain only *one* of the unknowns.

CASE II. *Factors of denominator all linear but some repeated.*—If a linear factor $px + q$ occurs r times in the denominator, one must assume r fractions corresponding to it; the denominators of these fractions are $(px + q)$, $(px + q)^2, \cdots, (px + q)^r$. Thus, if the factors of the

denominator are

$$(x-3)^2 \qquad \text{and} \qquad (x-1),$$

the corresponding partial fractions are

$$\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{x-1}$$

where A, B, and C are to be determined as in the previous case.

Example

Evaluate $\displaystyle\int \frac{x^2+6x-1}{(x-3)^2(x-1)}dx.$

Solution

Assume

$$\frac{x^2+6x-1}{(x-3)^2(x-1)} \equiv \frac{A}{(x-3)}+\frac{B}{(x-3)^2}+\frac{C}{x-1}$$
$$\equiv \frac{A(x-3)(x-1)+B(x-1)+C(x-3)^2}{(x-3)^2(x-1)}.$$

Equating numerators we have

$$x^2+6x-1 \equiv A(x-3)(x-1)+B(x-1)+C(x-3)^2.$$

Let $x=1$: $\quad 6=4C$ or $C=\frac{3}{2}$.
Let $x=3$: $\quad 26=2B$ or $B=13$.
Let $x=0$: $\quad -1=3A-B+9C$ or $A=-\frac{1}{2}$.

Hence,

$$\int\frac{(x^2+6x-1)dx}{(x-3)^2(x-1)} = -\frac{1}{2}\int\frac{dx}{x-3}+13\int\frac{dx}{(x-3)^2}+\frac{3}{2}\int\frac{dx}{x-1}$$
$$= -\frac{1}{2}\log(x-3)-\frac{13}{x-3}+\frac{3}{2}\log(x-1)+C$$
$$= \log\sqrt{\frac{(x-1)^3}{x-3}}-\frac{13}{x-3}+C.$$

CASE III. *Denominator contains some irreducible quadratic factors.*—Corresponding to each factor of the form ax^2+bx+c we assume a fraction of the form

$$\frac{Ax+B}{ax^2+bx+c}.$$

Example

Evaluate $\int \frac{(8x^2 + 3)dx}{(x^2 + x + 1)(x - 2)}$.

Partial Solution

Assume

$$\frac{8x^2 + 3}{(x^2 + x + 1)(x - 2)} \equiv \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 2}$$
$$\equiv \frac{Ax(x - 2) + B(x - 2) + C(x^2 + x + 1)}{(x^2 + x + 1)(x - 2)}.$$

Equating numerators we have

$$8x^2 + 3 \equiv Ax(x - 2) + B(x - 2) + C(x^2 + x + 1).$$

Let $x = 2$: $\quad 35 = 7C$ or $C = 5$.
Let $x = 0$: $\quad 3 = -2B + C$ or $B = 1$.
Let $x = 1$: $\quad 11 = -A - B + 3C$ or $A = 3$.

We have then

$$\int \frac{(8x^2 + 3)dx}{(x^2 + x + 1)(x - 2)} = \int \frac{(3x + 1)dx}{x^2 + x + 1} + \int \frac{5\,dx}{x - 2}.$$

PROBLEMS

1. $\int \frac{x\,dx}{x - 3}$.
2. $\int \frac{(2x + 5)dx}{x - 1}$.
3. $\int \frac{(7 - 2x)dx}{3x - 5}$.
4. $\int \frac{(x^3 + 1)dx}{x - 1}$.
5. $\int \frac{(x + 7)dx}{x^2 + 2x - 8}$.
6. $\int \frac{(3x + 4)dx}{x^2 + 5x + 6}$.
7. $\int \frac{(x^2 + x + 1)dx}{x^2 - 7x + 10}$.
8. $\int \frac{(x - 6)dx}{x^2 - x}$.
9. $\int \frac{(6x^2 - 23x + 9)dx}{x^3 - 4x^2 + 3x}$.
10. $\int \frac{dx}{x^3 - 3x^2 + 2x}$.
11. $\int \frac{(x^2 - 17x + 22)dx}{(x - 1)(x - 3)(x + 2)}$.
12. $\int \frac{(3x - 1)dx}{(x - 4)(2x + 1)(x - 1)}$.
13. $\int \frac{x^3dx}{(x + 1)(x^2 - 4)}$.
14. $\int \frac{(x^3 + x + 1)dx}{x(x - 1)(x - 2)(x - 3)}$.
15. $\int \frac{dv}{v^2 - a^2}$. *Formula* (15)
16. $\int \sec\theta\,d\theta = \int \frac{\cos\theta\,d\theta}{1 - \sin^2\theta}$. *Formula* (17)

HINT: Let $\sin\theta = x$.

17. $\int \csc\theta\, d\theta.$

18. $\int \frac{dx}{2e^x - 1}.$

19. $\int \frac{(x - 1 - 2x^2)dx}{(x-1)^2(x-3)}.$

20. $\int \frac{(3x+4)dx}{(x+2)^2(x-6)}.$

21. $\int \frac{dx}{x^2(x^2-4)}.$

22. $\int \frac{(x^5 - 2)dx}{x^4 - 2x^3}.$

23. $\int \frac{dx}{x^3 - 10x^2 + 33x - 36}.$

24. $\int \frac{(-3x^2 + 7x - 16)dx}{x^3 - 5x^2 + 7x - 3}.$

25. $\int \sec^3\theta\, d\theta = \int \frac{\cos\theta\, d\theta}{(1 - \sin^2\theta)^2}.$

HINT: Let $\sin\theta = x$.

26. $\int \csc^3\theta\, d\theta.$

27. $\int \frac{(x^2 + 9x + 29)dx}{(x-4)(x^2+2x+3)}.$

28. $\int \frac{(2x^2 + 6x - 1)dx}{x^3 + x^2 + x}.$

29. $\int \frac{(6x^3 - 19x^2 + 23x - 28)dx}{(x-1)(x-4)(x^2+x+4)}.$

30. $\int \frac{(15 - 5x + 10x^2 - x^3)dx}{x^2(x^2+5)}.$

31. $\int \frac{(5x^2 - 20x + 1)dx}{(x^2+4)(2x^2+x+1)}.$

32. $\int \frac{(5x^3 + 10x^2 - 4x - 43)dx}{(3x^2 - x + 7)(x^2 + x + 5)}.$

33. $\int \frac{dx}{x^3 - 8}.$

34. $\int \frac{dx}{x^4 - 16}.$

35. $\int \frac{(4x^3 + 23x^2 - 14x + 52)dx}{x^4 + 6x^3 + 14x^2 + 36x + 48}.$

CHAPTER XVIII

THE DEFINITE INTEGRAL

93. Area under a curve as the limit of a sum of rectangles. We consider now the problem of computing the area A bounded by the continuous curve whose equation is $y = f(x)$, the x-axis, and the ordinates at $x = a$ and $x = b$

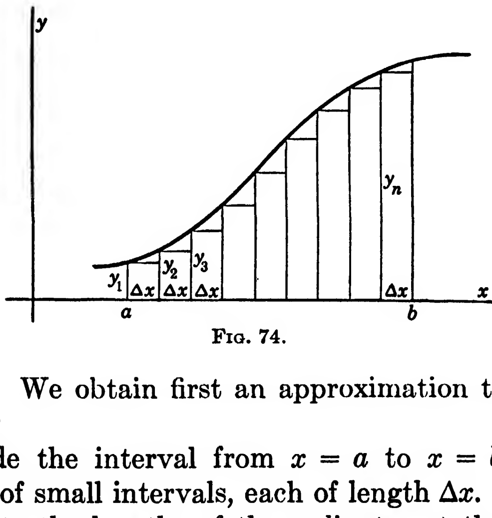

FIG. 74.

(Fig. 74). We obtain first an approximation to this area as follows:

1. Divide the interval from $x = a$ to $x = b$ into any number n of small intervals, each of length Δx.

2. Denote the lengths of the ordinates at the left-hand ends of these intervals by $y_1, y_2, y_3, \cdots y_n$.

3. Denote the sum of the areas of the rectangles shown in the figure by A'; then obviously,

$$A' = y_1\Delta x + y_2\Delta x + y_3\Delta x + \cdots + y_n\Delta x.$$

It is immediately clear that A' is a good approximation to the required area A if n is large. It is even evident that this approximation can be made *arbitrarily* good by taking n sufficiently large; *i.e.*, by taking a large enough number of rectangles one can get a value of A' which is arbitrarily

close to the required area A. This means of course that *the area A under the curve is exactly equal to the limit approached by A' as n increases without limit.* Using symbols,

$$A = \lim_{n \to \infty} A'$$

or

$$A = \lim_{n \to \infty} (y_1\Delta x + y_2\Delta x + y_3\Delta x + \cdots + y_n\Delta x).$$

The student must study the above equation and Fig. 74 until he has a very clear mental picture of the situation. Does it appear strange that even though each individual term in the parentheses approaches zero, the sum does not

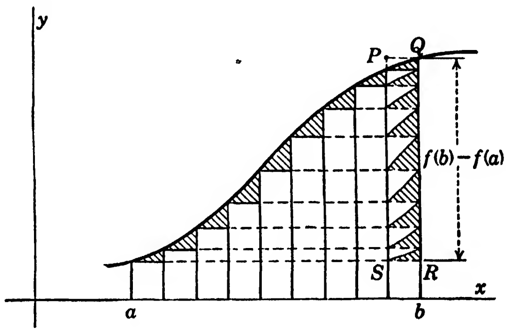

FIG. 75.

approach zero? Why not? Suppose that one takes a block of wood weighing 1 lb. and cuts it into two equal parts, then cuts each of these halves into two equal parts, and continues the process indefinitely. Does the weight of each piece approach zero? Does the sum of the weights of the pieces approach zero? How does this case differ from the one above?

The fact that the area A under the curve is equal to $\lim_{n \to \infty} A'$ is made still more obvious by Fig. 75. Clearly, the difference $A - A'$ consists of the sum of the little shaded areas on the tops of the rectangles. Projecting these little areas over into the last rectangle we see that their sum *is less than* the area of rectangle $PQRS$; *i.e.*,

$$A - A' < [f(b) - f(a)] \cdot \Delta x.$$

As $n \to \infty$ and $\Delta x \to 0$, the right-hand side of this inequality $\to 0$; *i.e.*,

$$A - A' \to 0, \qquad \text{or} \qquad A' \to A.$$

94. Notation.—For brevity we define the symbol

$$\sum_{i=1}^{n} y_i \Delta x$$

to stand for the *sum* of all terms which can be formed from $y_i \, \Delta x$ by letting the *index* i take successively all integral values from 1 to n inclusive; *i.e.*,

$$\sum_{i=1}^{n} y_i \Delta x \equiv y_1 \Delta x + y_2 \Delta x + y_3 \Delta x + \cdots + y_n \Delta x.$$

Using this convenient notation we may express the area under the curve in Fig. 74 as

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} y_i \Delta x.$$

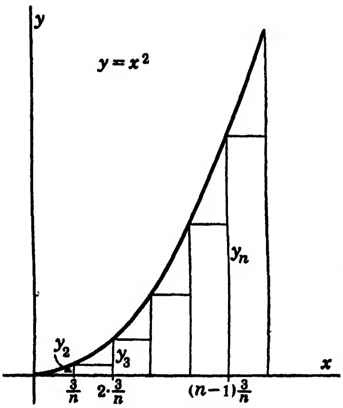

FIG. 76.

95. Computation of the limit. In order to determine the area under a curve by the method just discussed, one must express A' as a function of n and then try to determine the limit approached by A' as $n \to \infty$. The problem is very difficult in all but the simplest cases. The two following formulas, which can be proved by induction, will be useful in the problems of the next set.

(1) $\quad 1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}.$

(2) $\quad 1^3 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}.$

Example

Compute the area under the parabola $y = x^2$ from $x = 0$ to $x = 3$.

Solution (Fig. 76)

1. Dividing the interval from $x = 0$ to $x = 3$ into n equal parts, we have $\Delta x = 3/n$.
2. The ordinates are

$$y_1 = 0, \quad y_2 = \left(\frac{3}{n}\right)^2, \quad y_3 = \left(2 \cdot \frac{3}{n}\right)^2, \quad \cdots, \quad y_n = \left(\overline{n-1} \cdot \frac{3}{n}\right)^2.$$

3. $$\begin{aligned} A' &= \Delta x(y_1 + y_2 + y_3 + \cdots + y_n) \\ &= \frac{3}{n}\left[0 + \left(\frac{3}{n}\right)^2 + 2^2\left(\frac{3}{n}\right)^2 + 3^2\left(\frac{3}{n}\right)^2 + \cdots + (n-1)^2\left(\frac{3}{n}\right)^2\right] \\ &= \frac{27}{n^3}[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2]. \end{aligned}$$

4. $$\begin{aligned} A &= \lim_{n\to\infty} \frac{27}{n^3}[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2] \\ &= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right] \\ &= \lim_{n\to\infty} \frac{27}{6}\left[\frac{2n^3 - 3n^2 + n}{n^3}\right] \\ &= 9. \end{aligned}$$

PROBLEMS

1. Solve the preceding example using the ordinate at the right-hand side of each interval as the altitude of the corresponding rectangle instead of that at the left. First draw the figure carefully and show the rectangles.

2. Compute A' for the curve $y = x^2$ from $x = 0$ to $x = 3$ taking $n = 6$ and using the left-hand ordinate for each interval as the altitude of the corresponding rectangle; solve also using the right-hand ordinates. Compare these values of A' with the area A under the curve.

3. Compute A' for the curve $y = x^3$ from $x = 0$ to $x = 4$ taking $n = 8$. Use the left-hand ordinates as altitudes of the rectangles.

4. Compute the area A under the curve $y = x^3$ from $x = 0$ to $x = 4$ taking the left-hand ordinate for each interval as the altitude of the corresponding rectangle. Draw the figure and show the rectangles.

5. Solve Prob. 4 using the right-hand ordinates.

6. Express as a function of n the sum of the areas of n rectangles each with base Δx inscribed under the curve $y = x^4$ in the interval from $x = 0$ to $x = 4$. What does the limit of this sum represent? Try to compute this limit,

7. In Fig. 74, $f(x)$ is increasing throughout the interval from $x = a$ to $x = b$. Draw a corresponding figure in which $f(x)$ has a maximum point with horizontal tangent at $x = c$ where $a < c < b$. Show that in this case also $A = \lim_{n \to \infty} A'$. HINT: Draw the ordinate at $x = c$ and consider the two parts separately.

8. In what sense is the statement $A = \lim_{n \to \infty} A'$ really a *definition* of the area bounded by a curve? Compare it with the definition of the area of a circle as given in plane geometry.

96. Area under a curve by integration.—We shall now show that the area under a curve can be found by a much shorter process involving integration—and hence arrive at the remarkable fact that *the limit of a sum of the type*

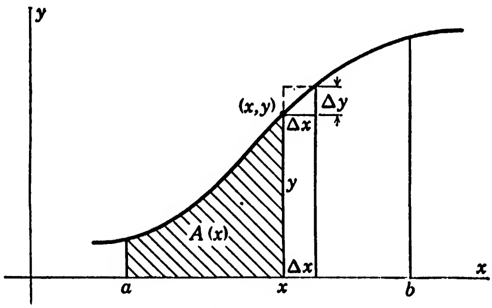

FIG. 77.

just discussed can be calculated by integration. We proceed as follows:

1. Think of the area under the curve shown in Fig. 77, as being generated by an ordinate of variable length which starts at $x = a$ and moves to the right, its upper end being always on the curve. The area generated when the moving ordinate has reached any point x is of course a function of x, and we may denote it by $A(x)$.

2. If now the ordinate moves a small additional distance Δx, the corresponding small area ΔA generated is clearly more than $y\,\Delta x$ and less than $(y + \Delta y)\Delta x$; *i.e.*,

$$y\,\Delta x < \Delta A < (y + \Delta y)\Delta x.$$

Dividing by Δx,

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y.$$

If now we let $\Delta x \to 0$, Δy also approaches 0 and, since $\lim\limits_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x} = \dfrac{dA}{dx}$, we have

$$\frac{dA}{dx} = y, \qquad \text{or} \qquad dA = y\,dx.$$

This equation states that for any position of the moving ordinate, the *rate* at which area is being added (in say square inches per inch moved in the x-direction) is equal to the value of y.

3. Since $dA/dx = y$, or $dA/dx = f(x)$, we can obviously find $A(x)$ by integration if we are careful to evaluate the constant of integration properly. Suppose that $\varphi(x)$ is an integral of $f(x)dx$; then

$$\begin{aligned} dA &= f(x)dx \\ A(x) &= \int f(x)dx \\ &= \varphi(x) + C. \end{aligned}$$

When $x = a$ the area generated is zero; hence,

$$0 = \varphi(a) + C, \qquad \text{or} \qquad C = -\varphi(a).$$

The area from $x = a$ to any other point x is then

$$A(x) = \varphi(x) - \varphi(a),$$

and the area from $x = a$ to $x = b$ is

$$A = \varphi(b) - \varphi(a).$$

We have then the following rule: If the curve $y = f(x)$ lies entirely on one side of the x-axis in the interval from $x = a$ to $x = b$, the area bounded by the curve, the x-axis, and the ordinates at $x = a$ and $x = b$, may be computed as follows:

1. *Find $\int f(x)dx$ in the usual way, obtaining a function $\varphi(x)$. The constant of integration may be omitted.*

2. *Find the value of $\varphi(x)$ when $x = b$ and the value when $x = a$, and subtract the latter from the former. This difference is equal to the required area.*

We shall use the symbol

$$\int_a^b f(x)dx$$

to denote the operation indicated in the two steps above; *i.e.*,

$$\int_a^b f(x)dx = \phi(x)\Big]_a^b = \phi(b) - \phi(a).$$

This symbol is read, "the integral from $x = a$ to $x = b$ of $f(x)dx$," and a and b are called the *lower limit* and the *upper limit* respectively of the integral.

Example 1

Compute the area under the curve $y = x^2$ from $x = 0$ to $x = 3$.

Solution

$$\begin{aligned} A = \int_0^3 y\,dx &= \int_0^3 x^2dx \\ &= \frac{x^3}{3}\Big]_0^3 \\ &= \frac{3^3}{3} - \frac{0^3}{3} = 9. \end{aligned}$$

If the scale on each axis is 1 in. = 1 unit, then the area under the curve is exactly 9 sq. in.

Example 2

Compute the area under one arch of the curve $y = \sin x$.

Solution

$$\begin{aligned} A &= \int_0^\pi \sin x\,dx \\ &= -\cos x\Big]_0^\pi \\ &= -\cos\pi - (-\cos 0) \\ &= -(-1) - (-1) = 2. \end{aligned}$$

PROBLEMS

1. Make a sketch similar to Fig. 77 but showing a function $f(x)$ which is *decreasing* throughout the interval from $x = a$ to $x = b$. Write out for this case the proof that $dA/dx = f(x)$.

2. Show that the difference between ΔA and $dA(= y\,dx)$ in Fig. 77 is an infinitesimal of higher order than Δx when $\Delta x \to 0$.

In each of the following cases, compute the area bounded by the given curve, the x-axis, and the given ordinates. Sketch the curve carefully and show the required area.

3. $y = x^2 + 1$; $x = 1$, $x = 4$.

4. $y = \dfrac{5}{x}$; $x = 1$, $x = 5$.

5. $y = 4 - x^2$; $x = -2$, $x = +2$.

6. $y = 5 + 4x - x^2$; $x = 0$, $x = 5$.

7. $y = \cos^2 x$; $x = 0$, $x = \frac{1}{2}\pi$.

8. $y = 6x - x^2$; $x = 0$, $x = 6$.

9. $y = 4x - x^3$; $x = 0$, $x = 2$.

10. $y = 2\sqrt{x}$; $x = 0$, $x = 9$.

11. $y = \log x$; $x = 1$, $x = e$.

12. $y = \dfrac{8}{x^2 + 4}$; $x = 0$, $x = 2$.

13. $y = x^3 - 7x^2 + 7x + 15$; $x = -1$, $x = 3$.

14. $y = \sin^3 x$; $x = 0$, $x = \frac{1}{2}\pi$.

15. $y = x\sqrt{16 - x^2}$; $x = 0$, $x = 4$.

16. $y = \dfrac{8x}{x^2 + 4}$; $x = 0$, $x = 2$.

In each of the following cases compute the area bounded by the two given curves. In order to do this, first sketch the curves carefully and find their points of intersection, then find the required area as the sum or difference of two areas.

17. $2y = x^2$ and $y^2 = 16x$.

18. $x^2 = 6y$ and $x^2 = 12y - 9$.

19. $y = 6x - x^2$ and $y = 2x$.

20. $x^2 = 4y$ and $y = \dfrac{8}{x^2 + 4}$.

21. Compute the area cut from the parabola $y = x^2 + 1$ by the chord joining the points $(-2, 5)$ and $(3, 10)$.

22. Sketch on the same axes the curve $y = (x - 3)^2(x + 1)$ and the line $y = x + 1$. Find their points of intersection. Compute each of the areas bounded by the curve and the line.

97. The fundamental theorem.—In computing the area under the curve in Fig. 74 we could have proceeded somewhat more generally as follows:

1. Divide the interval from $x = a$ to $x = b$ into n small intervals, *not necessarily equal,* and denote their lengths by $\Delta x_1, \Delta x_2, \Delta x_3, \cdots, \Delta x_n$ (Fig. 78).

2. Select *any* point x_1 in the first interval, any point x_2 in the second, $\cdots$, any point x_n in the last.

3. Form the sum

$$A' = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n.$$

This is obviously the sum of the areas of the n rectangles shown in Fig. 78 having Δx_1, Δx_2, $\cdots$, Δx_n as bases.

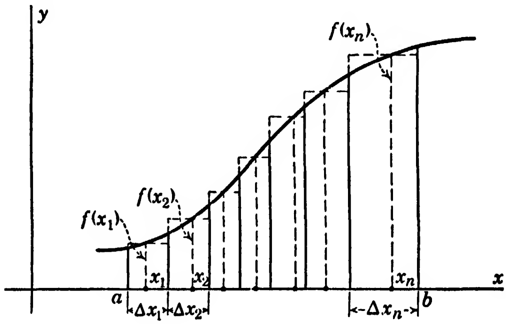

FIG. 78.

4. Then, if A is the area under the curve,

$$A = \lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} [f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n];$$

or

$$A = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

We have already shown, however, by an entirely independent method, that

$$A = \int_a^b f(x)dx.$$

Combining these two results we may state the important theorem known as the Fundamental Theorem of the integral calculus, namely:

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx.$$

We have confined our attention here to the case in which $f(x)$ is positive throughout the interval from $x = a$ to $x = b$, the area lying entirely above the x-axis. If the curve crosses the x-axis at $x = c$ where $a < c < b$, part of the area is above the x-axis and part is below. That the theorem holds also in this case can be shown by considering the parts from a to c and c to b separately.

98. The definite integral of a function over an interval.—Without thinking necessarily of the graph of the function or of the area under the curve, let us consider in an abstract way a function $f(x)$ which is single-valued and continuous for all values of x from a to b. Let us, then,

1. Divide the interval from $x = a$ to $x = b$ into n small intervals having lengths $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$ (Fig. 79).

FIG. 79.

2. Multiply the length of the first interval by the value of $f(x)$ at any point x_1 in this interval; multiply the length of the second interval by the value of $f(x)$ at any point x_2 in this interval, etc.

3. Add together the quantities so obtained; *i.e.*, form the sum

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n.$$

4. Consider the limit approached by this sum as the number of intervals used is increased indefinitely in such a way that the length of each one approaches zero; *i.e.*, consider

$$\lim_{\Delta x_i \to 0} [f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n] = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

The value of this limit, if it exists, is called the *definite integral* of the function $f(x)$ over the interval from $x = a$ to $x = b$.

In the next few chapters we shall see that many important problems of geometry, mechanics, etc., can be solved by expressing the required quantity as the limit of a sum of this type. Thus, we may express the length of a curve as the limit of a sum of n chords. We may also express the work done by a variable force, or the resultant force exerted by the water in a reservoir against the retaining dam, as the limit of such a sum. In each case, the limit can be evaluated by the use of the Fundamental Theorem. For, no matter what the physical aspects of the problem may be, this limit *may be interpreted* as representing the area under the curve $y = f(x)$ in the interval from $x = a$ to $x = b$ and, consequently,

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i \equiv \int_a^b f(x)dx.$$

99. Some properties of the definite integral. *A. The sign.*—If each of the quantities $f(x_i)\Delta x_i$ is positive, their sum is certainly positive and the limit of their sum is positive. If $a < b$, *i.e.*, if we integrate from left to right, Δx_i is positive. We have then

1. $\int_a^b f(x)dx$ *is* **positive** *if* $f(x)$ *is* **positive** *throughout the interval from a to b and* $a < b$. Thus, in Fig. 80,

$$\int_0^\pi \sin x \, dx = +2.$$

2. $\int_a^b f(x)dx$ *is* **negative** *if* $f(x)$ *is* **negative** *throughout the interval from a to b and* $a < b$. Thus, in Fig. 80,

$$\int_\pi^{\frac{3\pi}{2}} \sin x \, dx = -1.$$

3. *If* $f(x)$ *is positive over part of the interval and negative over part, then* $\int_a^b f(x)dx$ *represents the* **algebraic sum,** *not*

the arithmetic sum, of the areas above and below the x-axis. Thus, in Fig. 80,

$$\int_0^{\frac{3\pi}{2}} \sin x\, dx = +1.$$

When we speak of the area bounded by two curves, or by a curve, the x-axis, and the ordinates at $x = a$ and $x = b$, we shall always mean (unless otherwise specified) the area in the *arithmetic* sense. Thus, the area bounded by $y = \sin x$, $x = 0$, and $x = 3\pi/2$ is 3 square units. It cannot be found by integrating directly from 0 to $3\pi/2$, but must be

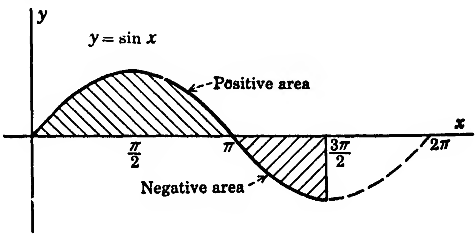

FIG. 80.

obtained by computing the two areas separately and adding.

B. Interchange of limits.—If the derivative of $\varphi(x)$ is $f(x)$ then,

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$$

while

$$\int_b^a f(x)dx = \varphi(a) - \varphi(b).$$

It is obvious, therefore, that

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

One may consequently interchange the limits on a definite integral if at the same time he changes the sign in front of the integral; thus,

$$\int_{\frac{\pi}{2}}^0 \sin x\, dx = -\int_0^{\frac{\pi}{2}} \sin x\, dx,$$

C. Subdividing the interval.—It is often desirable to divide the interval from a to b into two parts, one running from a to k and the other from k to b, where k is an intermediate point. The student can easily prove, using the method of part B, that

$$\int_a^b f(x)dx = \int_a^k f(x)dx + \int_k^b f(x)dx.$$

The theorem is true even if k is outside the interval from a to b. Thus,

$$\int_2^6 x^2dx = \int_2^4 x^2dx + \int_4^6 x^2dx.$$

Also

$$\int_2^6 x^2dx = \int_2^8 x^2dx + \int_8^6 x^2dx.$$
$$= \int_2^8 x^2dx - \int_6^8 x^2dx.$$

100. Change of limits corresponding to change of variable.—We have seen that in order to perform an integration one often has to make a change of variable. In the case of a definite integral, the necessity of changing the result back into terms of the original variable can be avoided by making a corresponding change in the limits.

Example

Evaluate $\int_0^4 \sqrt{16 - x^2}\, dx.$

Solution

Let $x = 4 \sin \theta.$

The integration is over the interval from $x = 0$ to $x = 4$. The corresponding interval for the new variable θ is obtained from the above equation; *i.e.*,

when $x = 4$,	$4 = 4 \sin \theta$,	$\sin \theta = 1$,	$\theta = \frac{1}{2}\pi.$
when $x = 0$,	$0 = 4 \sin \theta$,	$\sin \theta = 0$,	$\theta = 0.$

We have then,

$$\int_0^4 \sqrt{16 - x^2}\, dx = 16\int_0^{\frac{\pi}{2}} \cos^2 \theta\, d\theta = 4\pi.$$

101. Parametric equations.—The relation between x and y may be expressed in parametric form by equations such as

$$y = f(t), \qquad x = \varphi(t).$$

In this case, since $dx = \varphi'(t)dt$,

$$\int_a^b y\,dx = \int_{t_1}^{t_2} f(t)\varphi'(t)dt,$$

where the limits t_1 and t_2 are the values of t corresponding to $x = a$ and $x = b$; they are obtained from the equation $x = \varphi(t)$.

Example

Compute the area of the ellipse $x = a \cos \theta$, $y = b \sin \theta$.

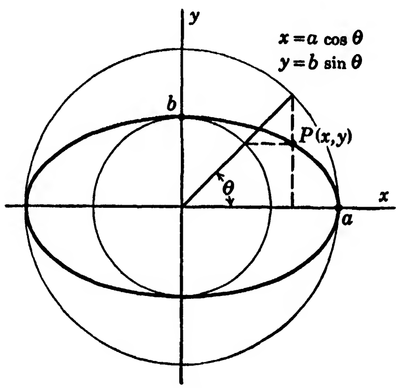

FIG. 81.

Solution (Fig. 81)

Taking advantage of the symmetry of the curve we may find the area in the first quadrant and multiply by 4; hence,

$$\begin{aligned} A &= 4\int_{x=0}^{x=a} y\,dx \\ &= 4\int_{\theta=\frac{\pi}{2}}^{\theta=0} (b \sin \theta)(-a \sin \theta)d\theta \\ &= 4ab\int_0^{\frac{\pi}{2}} \sin^2 \theta\, d\theta \\ &= \pi ab. \end{aligned}$$

PROBLEMS

1. Show that $\int_a^b f(x)dx = \int_c^k f(x)dx + \int_k^b f(x)dx.$

2. Evaluate $\int_{-2}^{+2}(x^3 - 4x)dx.$ Why is the result zero?

3. Why can one be certain that $\int_2^4 (x - 5)dx$ is negative without finding its value?

4. Under what conditions will $\int_a^b x^2dx$ be negative?

Evaluate each of the following definite integrals:

5. $\lim\limits_{\Delta x_i \to 0} \sum\limits_{i=1}^{n} x_i^4 \Delta x_i;\ x = 0$ to $x = 2.$

6. $\lim\limits_{\Delta x_i \to 0} \sum\limits_{i=1}^{n} (x_i^2 - 3x_i + 2)\Delta x_i;\ x = 1$ to $x = 3.$

7. $\lim\limits_{\Delta x_i \to 0} \sum\limits_{i=1}^{n} \sin^2 x_i \Delta x_i;\ x = 0$ to $x = \frac{1}{2}\pi.$

8. $\lim\limits_{\Delta x_i \to 0} \sum\limits_{i=1}^{n} x_i \log x_i \Delta x_i;\ x = 1$ to $x = e.$

9. $\lim\limits_{\Delta x_i \to 0} \sum\limits_{i=1}^{n} (\text{arc} \sin x_i)\Delta x_i;\ x = 0$ to $x = 1.$

10. $\int_0^1 (2 + \sqrt{x})^2 dx.$

11. $\int_0^1 xe^{-x^2}dx.$

12. $\int_0^4 \frac{6\,dx}{\sqrt[3]{8 - 2x}}.$

13. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^3 \theta\, d\theta.$

14. $\int_0^4 \frac{dx}{x^2 - 4x + 8}.$

15. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{\sin\theta \cos\theta}.$

16. $\int_0^{\frac{\pi}{4}} \tan^4 \theta\, d\theta.$

17. $\int_0^{\frac{\pi}{2}} x \cos 2x\, dx.$

18. $\int_0^{\frac{\pi}{8}} \frac{dx}{\cos^4 2x}.$

19. $\int_0^2 x^2\sqrt{4 - x^2}\, dx.$

20. $\int_1^2 \frac{\sqrt{x^2 - 1}}{x} dx.$

21. $\int_9^{16} \frac{\sqrt{x}\, dx}{x - 4}.$

22. $\int_1^6 \frac{x\, dx}{\sqrt{x + 3}}.$

23. $\int_0^3 x^3\sqrt{9 - x^2}\, dx.$

24. $\int_0^a \sqrt{a^2 - x^2}\, dx.$ **25.** $\int_4^7 \frac{x\, dx}{(x-3)^{\frac{3}{2}}}.$

26. $\int_0^1 \frac{x^3 dx}{\sqrt{x^2+1}}.$

27. Compute the area of a circle both with and without using the parametric equations.

28. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

29. Find the area bounded by the x-axis and one arch of the cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta).$

30. Compute the area of the hypocycloid whose equations are $x = a\cos^3\theta,\ y = a\sin^3\theta.$

31. Compute the area bounded by the parabola $y = x^2 - 4x$ and the line $y = x.$

32. Find the area bounded by the curve $y = x^3 + 3x^2$ and the line $y = 4x.$

102. Improper integrals.—It has been assumed up to this point that the interval from $x = a$ to $x = b$ is finite and that the integrand is continuous over the interval. It is frequently necessary to consider integrals in which these conditions are not both satisfied. Such integrals are called *improper* integrals.

We shall consider first the case in which the limits are not both finite. Values are assigned to integrals of this type by the following definitions:

$$(1)\quad \int_a^\infty f(x)dx \qquad \text{means} \qquad \lim_{h\to\infty}\int_a^h f(x)dx.$$

$$(2)\quad \int_{-\infty}^b f(x)dx \qquad \text{means} \qquad \lim_{h\to-\infty}\int_h^b f(x)dx.$$

$$(3)\quad \int_{-\infty}^\infty f(x)dx \qquad \text{means} \qquad \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx.$$

In case the limit involved does not exist, the integral is said not to exist. It is of course understood that the two integrals in definition (3) are to be evaluated in accordance with definitions (1) and (2). Hence, it is necessary that each of these exist separately in order that an integral of type (3) may have a value.

Example 1

Evaluate $\int_0^\infty \frac{dx}{(x+1)^2}$.

Solution

We shall integrate from 0 to h thus obtaining a function of h; then we shall examine the behavior of this function when $h \to \infty$.

$$\int_0^h \frac{dx}{(x+1)^2} = -\frac{1}{x+1}\Big]_0^h = -\frac{1}{h+1} + 1.$$

$$\lim_{h\to\infty}\left(1 - \frac{1}{h+1}\right) = 1.$$

Hence,

$$\int_0^\infty \frac{dx}{(x+1)^2} = 1.$$

The geometrical interpretation of the result is shown in Fig. 82. The area under the curve $y = \frac{1}{(x+1)^2}$ from $x = 0$ to $x = h$ is

$$1 - \frac{1}{h+1}.$$

As the point h moves indefinitely far to the right, this area continually increases and approaches 1 square unit as a limit.

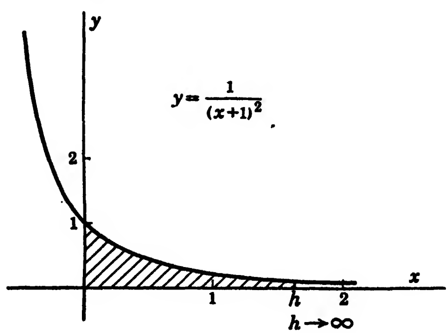

FIG. 82.

Example 2

Evaluate $\int_2^\infty \frac{dx}{x}$.

Solution

$$\int_2^h \frac{dx}{x} = \log x\Big]_2^h = \log h - \log 2.$$

$$\lim_{h\to\infty} (\log h - \log 2) = \infty.$$

Since the limit does not exist, the integral does not exist. The student should draw a figure and interpret this result geometrically.

The second kind of improper integral to be considered is that in which the function $f(x)$ has a vertical asymptote at one of the limits or between the limits. Values are assigned to such integrals by the following definitions, it being assumed that $a < b$:

(1) If $f(x)$ has a vertical asymptote at $x = a$, then

$$\int_a^b f(x)dx \qquad \text{means} \qquad \lim_{\epsilon\to 0} \int_{a+\epsilon}^b f(x)dx.$$

(2) If $f(x)$ has a vertical asymptote at $x = b$, then

$$\int_a^b f(x)dx \qquad \text{means} \qquad \lim_{\epsilon\to 0} \int_a^{b-\epsilon} f(x)dx.$$

(3) If $f(x)$ has a vertical asymptote at $x = c$, where $a < c < b$, then

$$\int_a^b f(x)dx \qquad \text{means} \qquad \int_a^c f(x)dx + \int_c^b f(x)dx.$$

If the corresponding limit does not exist, the integral does not exist. In definition (3) it is assumed that the two integrals are to be evaluated in accordance with definitions (1) and (2). Hence, these limits must exist separately in order that an integral of type (3) may have a value.

Example

Evaluate $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$.

Solution (Fig. 83)

The curve $y = 1/(x-1)^2$ has a vertical asymptote at $x = 1$. In accordance with definition (3) above, we must integrate from 0 to $1 - \epsilon$. thus obtaining a function of ϵ, and then find the limit of this function as

$\epsilon \to 0$. Next we must integrate from $1 + \epsilon$ to 2 and find the limit of this result as $\epsilon \to 0$. If the limits exist, their sum is the value of the given integral. If either or both of the limits are nonexistent, the integral has no value.

$$\int_0^{1-\epsilon} \frac{dx}{(x-1)^2} = \frac{1}{\epsilon} - 1.$$

$$\lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} - 1\right) = \infty.$$

It is evident, without considering the part from $1 + \epsilon$ to 2, that the integral does not exist.

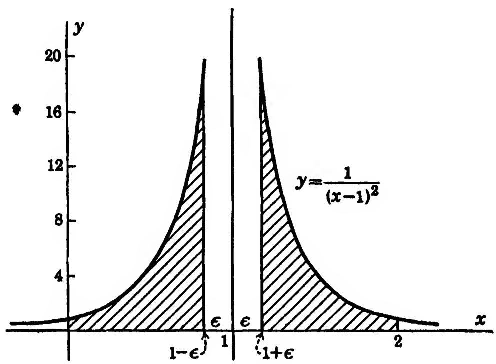

FIG. 83.

The student should observe that if we had proceeded carelessly, not noticing the discontinuity in the integrand, we should have obtained

$$\int_0^2 \frac{dx}{(x-1)^2} = -\frac{1}{x-1}\Big]_0^2 = -2.$$

This result is obviously incorrect. For we are integrating from left to right and the integrand is positive; the value of the integral would therefore have to be positive—if there were any value at all.

PROBLEMS

Evaluate the following improper integrals. Draw a figure for each.

1. $\int_1^\infty \frac{dx}{x}$.
2. $\int_1^\infty \frac{dx}{x^2}$.
3. $\int_0^\infty xe^{-x^2}dx$.
4. $\int_0^\infty \sin\theta\, d\theta$.
5. $\int_{-\infty}^0 e^x dx$.
6. $\int_0^1 \log x\, dx$.

7. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

8. $\displaystyle\int_2^\infty \frac{dx}{x(x-1)}$.

9. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

10. $\displaystyle\int_0^1 \frac{dx}{x}$.

11. $\displaystyle\int_{-1}^2 \frac{dx}{x^2}$.

12. $\displaystyle\int_{-3}^3 \frac{t\,dt}{\sqrt{9-t^2}}$.

13. $\displaystyle\int_0^{\frac{\pi}{2}} \tan\theta\,d\theta$.

14. $\displaystyle\int_0^9 \frac{dx}{\sqrt[3]{x-1}}$.

15. $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$.

16. $\displaystyle\int_a^{2a} \frac{dz}{\sqrt{z^2-a^2}}$.

17. $\displaystyle\int_0^a \frac{x^2dx}{\sqrt{a^2-x^2}}$.

18. $\displaystyle\int_4^\infty \frac{dx}{x\sqrt{x^2-4}}$.

19. $\displaystyle\int_0^{2a} \frac{dx}{(x-a)^2}$.

20. $\displaystyle\int_0^4 \frac{dx}{\sqrt{8x-x^2}}$.

21. Compute the area "bounded" by the witch $y = \dfrac{8a^3}{x^2+4a^2}$ and its asymptote.

22. Compute the area "bounded" by the curve $y^2(4-x) = x^3$ and the line $x = 4$.

CHAPTER XIX

DUHAMEL'S PRINCIPLE
APPLICATIONS TO GEOMETRY

103. Introduction.—It is probably intuitively evident to the student that in computing the area A under the curve in Fig. 84, it makes no difference whether we use the small rectangles such as $PQRS$, the large rectangles such as

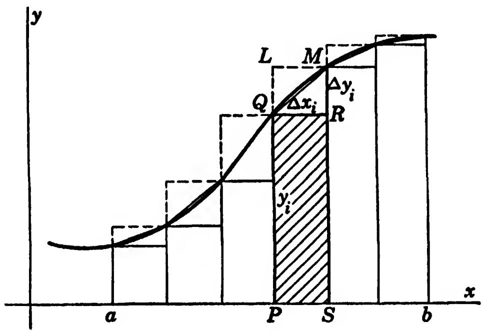

FIG. 84.

$PLMS$, or the trapezoids such as $PQMS$; *i.e.*, the sums

$$\sum_{i=1}^{n} y_i\Delta x_i, \qquad \sum_{i=1}^{n} y_{i+1}\Delta x_i, \qquad \text{and} \qquad \sum_{i=1}^{n} \tfrac{1}{2}(y_i + y_{i+1})\Delta x_i$$

all approach exactly the *same* limit when n is increased indefinitely in such a way that each $\Delta x_i \to 0$. The sum of the trapezoids shown in the figure is obviously a *better approximation* to A than that of either set of rectangles; but nothing would be gained by using the trapezoids if one were going to compute the *limit* of the sum. The result would be the same and the work might be more difficult.

Suppose we denote by $\Delta A_1, \Delta A_2, \cdots, \Delta A_n$, the parts or *elements* into which the required area A is subdivided by

the ordinates shown; *i.e.*, let us denote by ΔA_i the area *under the curve* in the ith strip. Obviously the area $PQRS$ or $PLMS$ or $PQMS$ may be taken as an *approximation* to ΔA_i. Furthermore—and this is very important—*the approximation differs from ΔA_i only by an infinitesimal of higher order than ΔA_i.* Thus, the difference between ΔA_i and $y_i\Delta x_i$ is less than $\Delta x_i \Delta y_i$ which is itself of higher order. In this connection the student should review the work on relative order of infinitesimals in Chap. XIII.

Consider now the two sums.

$$\sum_{i=1}^{n} \Delta A_i \equiv \Delta A_1 + \Delta A_2 + \Delta A_3 + \cdots + \Delta A_n;$$

$$\sum_{i=1}^{n} y_i \Delta x_i \equiv y_1 \Delta x_1 + y_2 \Delta x_2 + y_3 \Delta x_3 + \cdots + y_n \Delta x_n.$$

The first sum is identically equal to the area A under the curve for all values of n; consequently the limit of this sum is of course A. The second sum is only an approximation to A—but it is a special kind of approximation. *Each term in this second sum differs from the corresponding term in the first sum only by an infinitesimal of higher order.* It is the important theorem of Duhamel which assures us that under these conditions the *limit* of the second sum is exactly the same as the limit of the first. The theorem also assures us that the limit of the sum of the trapezoids shown would be exactly the same as that of the small rectangles, because each trapezoid differs from the corresponding rectangle by an amount $\frac{1}{2}\,\Delta x_i \Delta y_i$ which is of higher order than $y_i \Delta x_i$.

The formal statement of the theorem is as follows: *Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$, is a set of n positive infinitesimals such that*

$$\lim_{\substack{\alpha_i \to 0 \\ n \to \infty}} (\alpha_1 + \alpha_2 + \cdots + \alpha_n) = L.$$

Suppose that $\beta_1, \beta_2, \cdots \beta_n$ is another set of n positive infinitesimals and that each β_i differs from the corresponding

α_i by an infinitesimal of higher order than α_i. Then

$$\lim_{\substack{\beta_i \to 0 \\ n \to \infty}} (\beta_1 + \beta_2 + \cdots + \beta_n) = L.$$

This theorem, the proof of which will be omitted, is true also if all of the infinitesimals are negative. If some of them are positive and some negative it is true provided certain other conditions are satisfied. These conditions are usually satisfied and we shall proceed without mentioning them.

For use in the subsequent applications we shall in the next article formulate the above theorem in a slightly different form under the name of *Duhamel's principle.*

104. Duhamel's principle.—*Suppose that a quantity Q whose magnitude is to be calculated, can be subdivided into n parts or elements, ΔQ_1, ΔQ_2, $\cdots$, ΔQ_n, such that*

$$Q \equiv \Delta Q_1 + \Delta Q_2 + \cdots + \Delta Q_n.$$

Suppose that each element ΔQ_i can be approximated by an infinitesimal α_i so that for every i the difference $|\Delta Q_i - \alpha_i|$ is an infinitesimal of higher order than ΔQ_i. Then the limit approached by $\sum_{i=1}^{n} \alpha_i$ when n is increased indefinitely in such a way that each $\alpha_i \to 0$ is exactly the quantity Q.

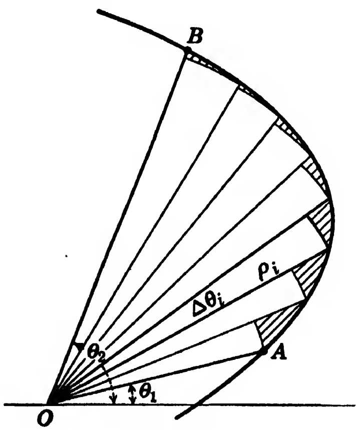

FIG. 85.

The usefulness of the principle lies in the possibility of choosing the α_i so that the limit of their sum can be found by integration; *i.e.*, by the use of the Fundamental Theorem. The next article gives an immediate illustration.

105. Area in polar coordinates.—To illustrate the foregoing theorem let us try to determine the area bounded by the polar curve $\rho = f(\theta)$ and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ (Fig. 85).

1. By dividing the angle $\theta_2 - \theta_1$ into n parts, $\Delta\theta_1$, $\Delta\theta_2$, $\cdots$, $\Delta\theta_n$, and drawing the radii vectors, the required area A is subdivided into n parts, ΔA_1, ΔA_2, $\cdots$, ΔA_n, and of course

$$A = \Delta A_1 + \Delta A_2 + \cdots + \Delta A_n.$$

2. Each ΔA_i may be approximated by the area of the *circular sector* $\frac{1}{2}\rho_i^2\Delta\theta_i$. The proof that the error is of higher order is left as an exercise in the next set.

3. The required area A is then exactly equal to the value of

$$\lim_{\Delta\theta_i\to 0}\sum_{i=1}^{n}\tfrac{1}{2}\rho_i^2\Delta\theta_i = \tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2 d\theta.$$

Example

Compute the entire area bounded by the curve $\rho = a\sin 3\theta$.

Solution (Fig. 86)

We may find the area of one "petal" and multiply by 3, *i.e.*,

$$\begin{aligned}A &= 3\cdot\tfrac{1}{2}\int_0^{\frac{\pi}{3}}\rho^2 d\theta\\ &= \frac{3a^2}{2}\int_0^{\frac{\pi}{3}}\sin^2 3\theta\, d\theta\\ &= \frac{3a^2}{4}\int_0^{\frac{\pi}{3}}(1-\cos 6\theta)d\theta\\ &= \frac{\pi a^2}{4}.\end{aligned}$$

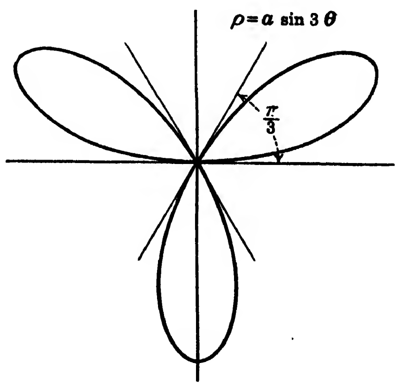

FIG. 86.

PROBLEMS

1. Show that area $QLMR$ (Fig. 84) is an infinitesimal of higher order than area $PQRS$ when $\Delta x_i \to 0$. Hence, show that the difference between ΔA_i and $y_i\Delta x_i$ is of higher order.

2. Show that in Fig. 85 the difference between ΔA_i and $\frac{1}{2}\rho_i^2\Delta\theta_i$ is an infinitesimal of higher order than ΔA_i. HINT: Note that the value of ΔA_i is between that of $\frac{1}{2}\rho_i^2\Delta\theta_i$ and $\frac{1}{2}(\rho_i + \Delta\rho_i)^2\Delta\theta_i$ and that the difference between these is of higher order.

3. Sketch the curve $\rho = a \sin 3\theta$ and shade the area which one would obtain by evaluating $\frac{1}{2}\int_0^{\frac{\pi}{2}} \rho^2 d\theta$.

4. Under what conditions would the value of $\int_{\theta_1}^{\theta_2} \frac{1}{2}\rho^2 d\theta$ be negative? HINT: Consider the sum $\Sigma \frac{1}{2}\rho_i^2 \Delta\theta_i$ and note that ρ_i^2 is positive for all real values of ρ.

5. Sketch the cardioid $\rho = a(1 - \cos\theta)$. State which of the following integrals would give its entire area: $4\int_0^{\frac{\pi}{2}} \frac{1}{2}\rho^2 d\theta$; $2\int_0^{\pi} \frac{1}{2}\rho^2 d\theta$; $\int_0^{2\pi} \frac{1}{2}\rho^2 d\theta$.

6. Sketch the curve $\rho = a \cos 3\theta$. State which of the following integrals would give its entire area: $6\int_0^{\frac{\pi}{6}} \frac{1}{2}\rho^2 d\theta$; $\int_0^{\pi} \frac{1}{2}\rho^2 d\theta$; $\int_0^{2\pi} \frac{1}{2}\rho^2 d\theta$; $2\int_0^{\frac{\pi}{2}} \frac{1}{2}\rho^2 d\theta$.

Sketch each of the following curves and find the entire area which it bounds:

7. $\rho = 8\cos\theta$.
8. $\rho = a(1 + \cos\theta)$.
9. $\rho = a \sin 2\theta$.
10. $\rho = 4\cos 3\theta$.
11. $\rho = a\sin 3\theta$.
12. $\rho^2 = a^2 \cos 2\theta$.
13. $\rho^2 = a^2 \sin 2\theta$.
14. $\rho = 3 - \sin\theta$.
15. $\rho^2 = 2\sin\theta - 1$.

16. Find the area of one loop of the curve $\rho = 6\cos 3\theta$.

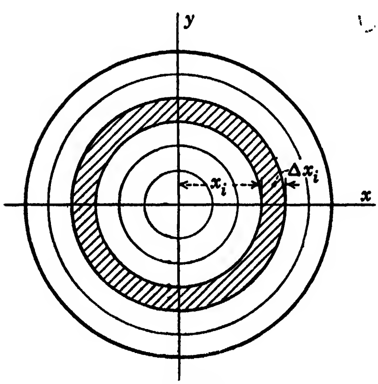

FIG. 87.

17. Find the area inside the small loop of the curve $\rho = 1 + 2\sin\theta$.

18. Find the area common to the two circles $\rho = a$ and $\rho = 2a\cos\theta$.

19. Find the area common to the circle $\rho = 3\sin\theta$ and the cardioid $\rho = \sin\theta + 1$.

20. Find the area which is inside the lemniscate $\rho^2 = 8\cos 2\theta$ and outside the circle $\rho = 2$.

21. Find the area bounded by the parabola $\rho \sin^2\theta = \cos\theta$ and the line $\rho\cos\theta = 1$.

22. Divide the circle $x^2 + y^2 = r^2$ into n concentric rings as indicated in Fig. 87 and denote by ΔA_i the area of the ith ring. Show that

$$|\Delta A_i - 2\pi x_i \Delta x_i|$$

is an infinitesimal of higher order than ΔA_i when $\Delta x_i \to 0$, and hence the area of the circle is given by

$$\int_0^r 2\pi x\, dx.$$

23. Using Duhamel's Principle, find the area bounded by the y-axis and the parabola $x = y^2 - 2y - 8$ by dividing it into strips parallel to the x-axis.

24. Set up integrals for the area bounded by the curves $x^2 + y^2 = 4$ and $y^2 = 3x$, using both vertical and horizontal strips.

106. Volumes of solids of revolution.—Suppose that the area bounded by the curve whose equation is $y = f(x)$, the x-axis, and the ordinates at $x = a$ and $x = b$, is revolved

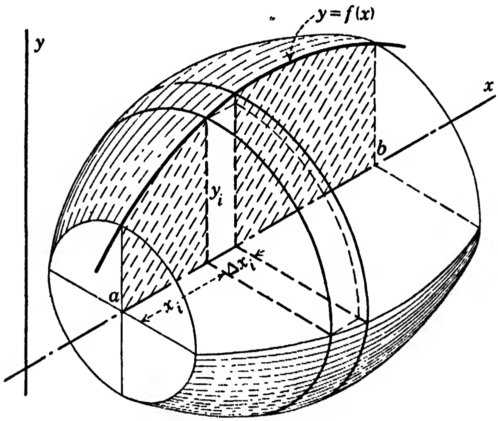

FIG. 88.

about the x-axis. The volume V of the solid generated can be found easily as follows (Fig. 88):

1. Denote by ΔV_i the volume generated by the area ΔA_i under the curve in the ith strip. Then, of course,

$$V = \sum_{i=1}^{n} \Delta V_i.$$

2. Each ΔV_i may be approximated by the volume of the corresponding cylinder generated by the area $y_i\Delta x_i$. This volume is $\pi y_i^2 \Delta x_i$. The proof that the difference is of

higher order is left as an exercise for the student in the next set.

3. The required volume V is then *exactly* equal to the value of

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \pi y_i^2 \Delta x_i = \int_a^b \pi y^2 dx.$$

The student should not use the above result as a formula but should draw the figure for each case and analyze the situation carefully. The axis of revolution may of course be a line other than the x-axis.

Example

Compute the volume of the solid generated by revolving the area bounded by the y-axis and the parabola $y^2 - 4y + 2x - 5 = 0$ about the y-axis.

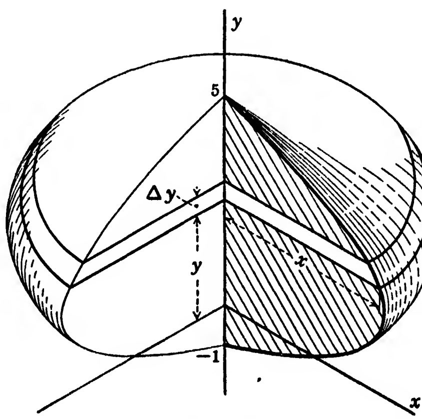

FIG. 89.

Solution (Fig. 89)

Taking the interval from $y = -1$ to $y = 5$ and proceeding as indicated above, the student will readily see that the required volume is given by

$$\int_{-1}^{5} \pi x^2 dy = \frac{\pi}{4}\int_{-1}^{5} (5 + 4y - y^2)^2 dy = \frac{324\pi}{5}.$$

107. Cylindrical shell method.—The volume of a solid of revolution may sometimes be calculated more easily by

dividing it up into cylindrical shells. The method is illustrated by the following example.

Example

Compute the volume of the solid generated by revolving the area bounded by the x-axis and one arch of the curve $y = \sin x$, about the y-axis.

Solution (Fig. 90)

Denote by ΔV_i the volume generated by the area ΔA_i under the curve in the ith strip. Then ΔV_i is approximately equal to the volume of a cylindrical shell with inside radius x_i, height y_i, and thickness Δx_i.

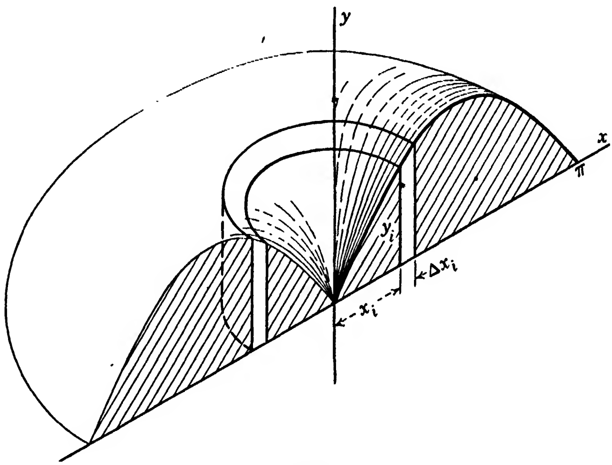

FIG. 90.

This in turn is approximately $2\pi x_i y_i \Delta x_i$. The student can easily show that the difference is of higher order. The required volume is then given by

$$\begin{aligned}\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} 2\pi x_i y_i \Delta x_i &= \int_0^{\pi} 2\pi xy\, dx \\ &= 2\pi \int_0^{\pi} x \sin x\, dx \\ &= 2\pi[-x \cos x + \sin x]_0^{\pi} \\ &= 2\pi^2.\end{aligned}$$

PROBLEMS

1. Show that in Fig. 88 the difference between $\pi y_i^2 \Delta x_i$ and ΔV_i is of higher order. HINT: If $f(x)$ is either increasing or decreasing throughout

an interval Δx_i, the value of ΔV_i is between $\pi y_i^2 \Delta x_i$ and $\pi(y_i + \Delta y_i)^2 \Delta x_i$; show that the difference between these two is of higher order. What about an interval in which $f(x)$ is not everywhere increasing or decreasing?

2. Show that in Fig. 90 the difference between ΔV_i and $2\pi x_i y_i \Delta x_i$ is of higher order. HINT: The value of ΔV_i is between $2\pi x_i y_i \Delta x_i$ and $2\pi(x_i + \Delta x_i)(y_i + \Delta y_i)\Delta x_i$.

3. Compute the volume of the sphere generated by revolving a circle of radius r about a diameter.

4. Derive the formula for the volume of a cone. HINT: Revolve the area bounded by $y = 0$, $x = h$, and $y = rx/h$ about the x-axis.

5. Compute the volume of a parabolic reflector which is 8 in. in diameter and 6 in. deep.

6. Compute the volume of the solid generated by revolving one arch of the sine curve about the x-axis.

7. Find the volume of the ellipsoid generated by revolving the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the x-axis. The y-axis.

8. Compute the volume of the solid generated by revolving the area bounded by the curve $y = \dfrac{8}{x^2 + 4}$, the x-axis, and the lines $x = \pm 2$, about the y-axis.

9. The area bounded by the curve $y = \tan x$, the x-axis, and the line $x = \pi/4$ is revolved about the x-axis. Find the volume generated.

10. The area bounded by the x-axis and the parabola $y = 4x - x^2$ is revolved about the y-axis. Find the volume generated.

11. The area bounded by the coordinate axes and the curve $y = \cos x$ $(0 \leq x \leq \frac{1}{2}\pi)$ is revolved about the y-axis. Set up integrals for the volume generated using both methods. Complete the solution using the simpler integral.

12. The area "bounded" by the coordinate axes and the curve $y = e^{-x}$ is revolved about the x-axis. Find the volume generated. Note that the integral is improper.

13. Solve Prob. 12 for the case in which the area is revolved about the y-axis.

14. Compute the volume of the solid generated by revolving the area bounded by one arch of the curve $y = 2 \sin x$ and the line $y = 1$ about the line $y = 1$. About the x-axis.

15. The area bounded by the x-axis, the curve $y = \log x$, and the line $x = e$, is revolved about the y-axis. Compute the volume of the solid generated. Use both methods

16. The area common to the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 8x$ is revolved about their common chord. Compute the volume of the solid generated.

17. Compute the volume of the solid generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

18. Solve Prob. 17 for the case in which the area is revolved about the y-axis. Use the cylindrical shell method.

19. The area "bounded" by the curve $y = \dfrac{8}{x^2 + 4}$ and the x-axis is revolved about the x-axis. Compute the volume generated.

20. The area above the x-axis bounded by the circle $x^2 + y^2 = r^2$, and the lines $y = x$ and $y = -x$, is revolved about the y-axis. Compute the volume generated. Note that this is the volume common to a sphere and a cone. Use the cylindrical shell method.

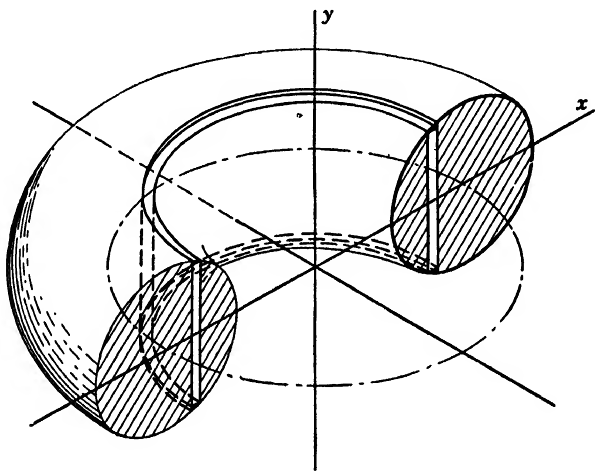

FIG. 91.

21. Compute the volume bounded by the xy-plane, the paraboloid $x^2 + y^2 = az$, and the cylinder $x^2 + y^2 = a^2$. Use both methods.

22. Compute the volume common to the sphere $x^2 + y^2 + z^2 = 6$ and the paraboloid $x^2 + y^2 = z$. HINT: Revolve the area common to the circle $x^2 + z^2 = 6$ and the parabola $z = x^2$ about the z-axis. Use cylindrical shell method.

23. The axis of a right circular cylinder of radius 8 in. passes through the center of a sphere of radius 16 in. What volume does the cylinder cut from the sphere?

24. Find the volume generated by revolving the circle $x^2 + y^2 = 2rx$ about the y-axis.

25. Compute the volume of the torus generated by revolving the circle $(x - R)^2 + y^2 = r^2$ (Fig. 91) about the y-axis. HINT: Use cylindrical shell method and after setting up the integral let $x - R = t$.

108. Volumes by parallel slicing.—The first of the above two methods for finding the volume of a solid of revolution can easily be generalized to include any solid for which the area of the cross section at a distance x from a fixed plane is a *known* function of x — say $A(x)$. For if the solid be cut into n slices by planes parallel to this fixed plane, the volume of the ith slice will be equal to $A(x_i)\Delta x_i$ except for an infinitesimal of higher order. Consequently,

$$V = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} A(x_i)\Delta x_i = \int_a^b A(x)dx.$$

Example 1

Compute the volume of the conoid shown in Fig. 92.

Solution

At a distance x from the yz-plane, the section is a triangle with base $2\sqrt{r^2 - x^2}$ and altitude h. We have, then,

$$A(x) = h\sqrt{r^2 - x^2},$$

and

$$V = \int_{-r}^{r} h\sqrt{r^2 - x^2}\, dx$$
$$= \tfrac{1}{2}\pi r^2 h.$$

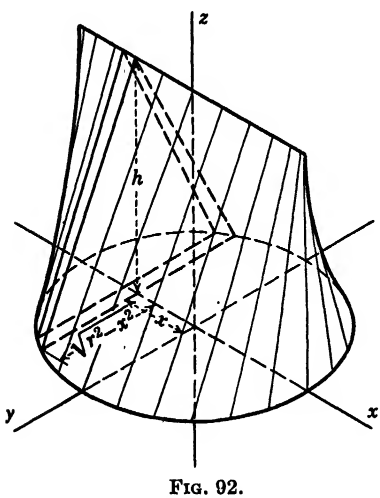

FIG. 92.

Example 2

The axes of two right circular cylinders of radius r intersect at right angles. Compute the volume common to the two cylinders.

Solution (Fig. 93)

From the figure it is clear that at a distance z above the xy-plane the cross section of the required volume is a *square* with each side equal to $2\sqrt{r^2 - z^2}$. Consequently,

$$A(z) = 4(r^2 - z^2),$$

and

$$V = 2\int_0^r 4(r^2 - z^2)dz$$
$$= \tfrac{16}{3}r^3.$$

PROBLEMS

1. A right pyramid has a rectangular base 12 by 8 in. Its altitude is 16 in. What is the area of a section at a distance z above the base? What is its volume?

2. Find the volume bounded by the three coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

3. A solid has the circle $x^2 + y^2 = 16$ for a base. The sections parallel to the yz-plane are squares. Sketch the solid and compute its volume.

4. Find the volume of an elliptical conoid whose base is the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and whose altitude is 6.

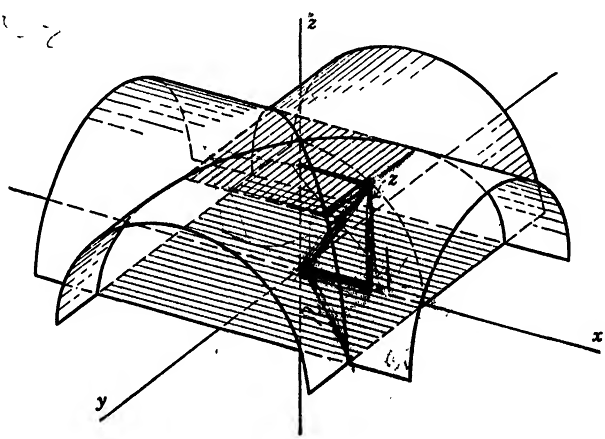

FIG. 93.

5. Compute the volume bounded by the paraboloid $z = x^2 + 4y^2$ and the plane $z = 2$.

6. Compute the volume bounded by the paraboloid $y - 4x^2 - 9z^2 = 0$ and the plane $y = 6$.

7. Compute the volume common to the cylinders $x^2 + y^2 = r^2$ and $x^2 + z^2 = r^2$. Draw the figure carefully.

8. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

9. A wedge is cut from a tree 6 ft. in diameter by first making a horizontal saw cut half way through the trunk and then making a second cut inclined at 45° to the horizontal and meeting the first one along a diameter of the section. Compute the volume of the wedge.

10. Find the volume of the wedge cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = a + x$.

11. The diameter of a solid right circular cylinder is 8 in. and its height is 12 in. The cylinder is cut into two parts by a plane which passes through a diameter of the lower base and is tangent to the upper base. Show that the volume of the smaller piece is 128 cu. in.

12. Find the volume cut from the paraboloid $x^2 + 4y^2 + z = 16$ by the plane $z = 12$.

109. Length of a curve.—The length s of the arc AB in Fig. 94 is *defined* as the limit approached by the sum of the lengths of the n chords indicated when n increases

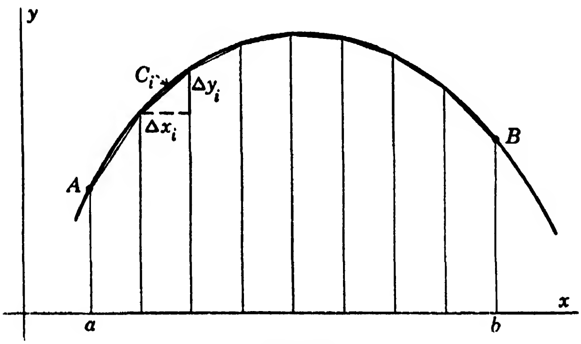

FIG. 94.

indefinitely in such a way that the length of each chord approaches zero. The length of the ith chord is

$$C_i = \sqrt{\Delta x_i^2 + \Delta y_i^2};$$

hence,

$$s = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

In order to express this limit as an integral we may first, in each term of the sum, replace $\Delta x^2 + \Delta y^2$ by $dx^2 + dy^2$. This may be interpreted geometrically as replacing each chord by the corresponding segment of tangent line as indicated to an enlarged scale in Fig. 95. It can be shown that the difference is of higher order. We have, then,

$$s = \lim_{dx_i \to 0} \sum_{i=1}^{n} \sqrt{dx_i^2 + dy_i^2}$$

or

$$s = \lim_{dx_i \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{dy_i}{dx_i}\right)^2}\, dx_i.$$

Using the Fundamental Theorem, we may then write this last result in the form

$$\text{(I)} \qquad s = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The student may show that an alternate form of the result, which may be used when it is more convenient to regard y as the independent variable, is

$$\text{(II)} \qquad s = \int_{y=c}^{y=d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Finally, a third form, which is particularly convenient when x and y are expressed parametrically as functions of a third variable t, is

$$\text{(III)} \qquad s = \int \sqrt{dx^2 + dy^2}.$$

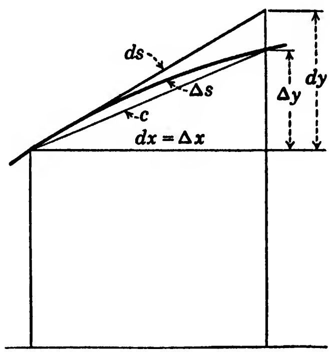

FIG. 95.

The length $ds = \sqrt{dx^2 + dy^2}$ in Fig. 95 is called the *differential of arc.* It is the principal part of the increment Δs in the arc corresponding to an increment Δx in x. The student can probably see intuitively that ds is a good approximation to Δs if Δx is small—just as dy is a good approximation to Δy. It can be shown that $\lim_{\Delta x \to 0} \frac{ds}{\Delta s} = 1$, which means of course that the difference between ds and Δs is an infinitesimal of higher order than either when $\Delta x \to 0$.

The corresponding formula for length of arc in polar coordinates can be obtained by applying to (III) the usual transformation to polar coordinates. Letting

$$x = \rho \cos \theta \qquad \text{and} \qquad y = \rho \sin \theta,$$

and remembering that both ρ and θ are variables, we have

$$dx = -\rho \sin\theta\, d\theta + \cos\theta\, d\rho$$
$$dy = \rho\cos\theta\, d\theta + \sin\theta\, d\rho.$$

The result of substituting these expressions for dx and dy in (III) is

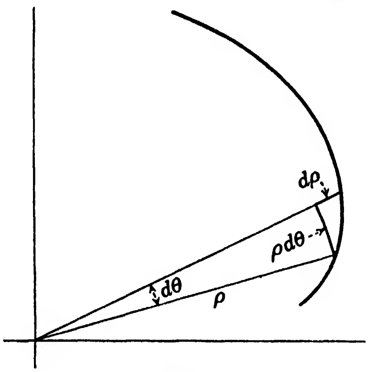

FIG. 96.

(IV) $\quad s = \int\sqrt{d\rho^2 + \rho^2 d\theta^2}.$

If θ is to be regarded as the independent variable this may be put into the convenient form

(V) $\quad s = \displaystyle\int_{\theta_1}^{\theta_2}\sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta.$

An easy method of remembering formula (IV) is indicated in Fig. 96. It shows that a small element of arc is approximately equal to the hypotenuse of a right triangle with one side equal to $\rho\, d\theta$ and the other equal to $d\rho$.

Example 1

Compute the length of the parabola $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$.

Solution

$$y = \tfrac{1}{2}x^2$$
$$\frac{dy}{dx} = x;$$
$$s = \int_0^1\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1\sqrt{1 + x^2}\, dx$$
$$= \int_0^{\frac{\pi}{4}}\sec^3\theta\, d\theta \qquad \text{(letting } x = \tan\theta)$$
$$= \tfrac{1}{2}[\sec\theta\tan\theta + \log(\sec\theta + \tan\theta)]_0^{\frac{\pi}{4}}$$
$$= \tfrac{1}{2}[\sqrt{2} + \log(\sqrt{2} + 1)]$$
$$= 1.114.$$

Example 2

Compute the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

Solution

$$dx = a(1 - \cos\theta)d\theta; \qquad dy = a \sin\theta d\theta.$$

$$\begin{aligned}\sqrt{dx^2 + dy^2} &= a\sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\, d\theta \\ &= a\sqrt{2 - 2\cos\theta}\, d\theta \\ &= 2a \sin \tfrac{1}{2}\theta\, d\theta.\end{aligned}$$

$$s = \int_0^{2\pi} 2a \sin \tfrac{1}{2}\theta\, d\theta = 8a.$$

Example 3

Compute the perimeter of the circle $\rho = 2a \sin\theta$.

Solution (Fig. 97)

$$\frac{d\rho}{d\theta} = 2a \cos\theta.$$

$$s = \int_0^{\pi} \sqrt{4a^2 \sin^2\theta + 4a^2 \cos^2\theta}\, d\theta$$

$$= 2a\int_0^{\pi} d\theta = 2\pi a.$$

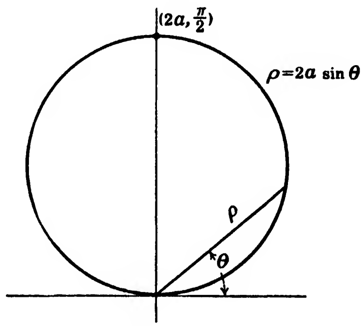

FIG. 97.

110. Areas of surfaces of revolution.—Suppose that the curve together with the chords shown in Fig. 98 is revolved about the x-axis. The curve generates a surface of revolution while each chord generates the lateral surface of a

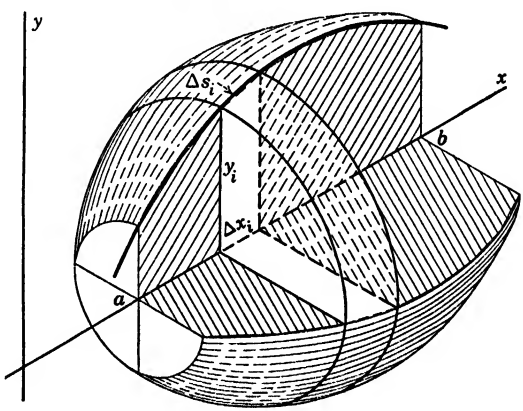

FIG. 98.

frustum of a cone. The area of the surface is defined as the limit approached by the sum of the lateral areas of the

frustums when $\Delta x_i \to 0$ and $n \to \infty$. The lateral area of the ith frustum is, if C_i is the length of chord,

$$2\pi(y_i + \tfrac{1}{2}\Delta y_i) \cdot C_i$$

which differs by an infinitesimal of higher order from

$$2\pi y\, ds_i.$$

It follows that the area of the surface of revolution is given by the value of

$$A = \int_{x=a}^{x=b} 2\pi y\, ds.$$

The details of the derivation are left to the student.

Example

Compute the surface area of the sphere generated by revolving the upper half of the circle $x^2 + y^2 = r^2$ about the x-axis.

Solution

$$\frac{dy}{dx} = -\frac{x}{y}; \qquad ds = \sqrt{1 + \frac{x^2}{y^2}}\, dx = \frac{r}{y} dx.$$

$$A = \int_{x=-r}^{x=r} 2\pi y\, ds$$

$$= 2\pi \int_{-r}^{r} y \cdot \frac{r}{y} dx = 2\pi r \int_{-r}^{r} dx = 4\pi r^2.$$

PROBLEMS

1. Show that the difference between ds and c (Fig. 95) is of higher order than Δx. HINT: Show that $\lim\limits_{\Delta x \to 0} \dfrac{\sqrt{dx^2 + dy^2}}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}} = 1.$

2. Compute the length of the circle $x^2 + y^2 = r^2$. Is this a legitimate derivation of the formula $C = 2\pi r$, or are you assuming the formula in arriving at your result?

3. Compute the length of the curve $y^2 = x^3$ from the origin to the point where $x = \frac{7}{3}$.

4. Find the length of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from $(0, a)$ to (x_1, y_1).

5. Find the length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

6. Solve Prob. 5 using the parametric equations of the hypocycloid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

7. Find the length of the involute $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. Sketch the curve.

8. Show directly from Fig. 99 that in polar coordinates $ds = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,d\theta$. HINT: First compute C^2 as the hypotenuse of a right triangle, obtaining

$$C^2 = 2\rho^2(1 - \cos\Delta\theta) + \overline{\Delta\rho}^2 + 2\rho\,\Delta\rho(1 - \cos\Delta\theta).$$

Then find $\lim\limits_{\Delta\theta\to 0}\left(\frac{C}{\Delta\theta}\right)^2$ which is equal to $(ds/d\theta)^2$. Remember that $1 - \cos\Delta\theta = 2\sin^2\frac{1}{2}\Delta\theta$.

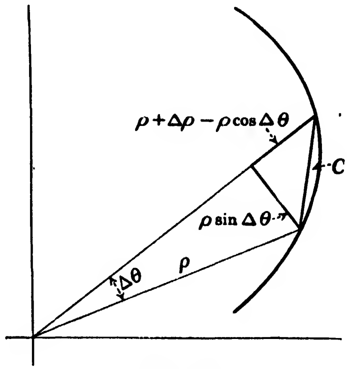

FIG. 99.

9. Find the perimeter of the cardioid $\rho = 4(1 + \cos\theta)$.

10. Suppose that the arc AB of the curve $\rho = f(\theta)$ (Fig. 100) is divided into n parts and circular arcs are drawn as indicated. Is the sum $\sum\limits_{i=1}^{n} \rho_i\Delta\theta_i$ an arbitrarily good approximation to the length of arc AB if each $\Delta\theta$ is sufficiently small? Is the difference between $\rho_i\Delta\theta_i$ and Δs_i an infinitesimal of higher order? Would you expect the value of $\int_{\theta_1}^{\theta_2}\rho\,d\theta$ to be the length of the arc?

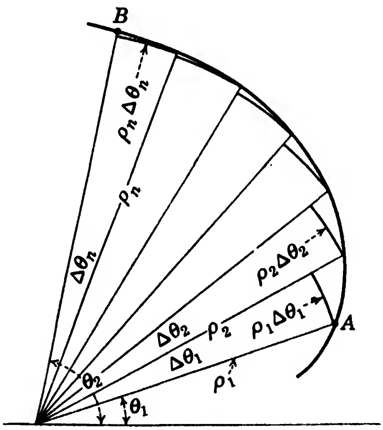

FIG. 100.

11. Write out in detail the derivation of the formula

$$A = \int_{y=c}^{y=d} 2\pi x\,ds$$

for the area of the surface generated when a curve is revolved about the y-axis.

12. Compute the area of the surface generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

13. Compute the area of the surface generated by revolving the arc of the parabola $y = x^2$ from $x = 0$ to $x = 2$ about the y-axis.

14. The parabolic reflector of an automobile headlight is 12 in. in diameter and 4 in. deep. What is its area?

15. Compute the lateral area of the cone generated by revolving the segment of the line $y = rx/h$ from $x = 0$ to $x = h$ about the x-axis.

16. Compute the surface area of the torus generated by revolving the circle $(x - R)^2 + y^2 = r^2$, $R > r$, about the y-axis.

17. Find the area of the surface generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

18. Find the area of the surface generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

19. Compute the area of the surface generated by revolving the part of the curve $y = \log x$ from $x = 0$ to $x = 1$ about the y-axis.

20. Sketch the circle $\rho = 2r \sin\theta$. Find the area of the surface generated when it is revolved about the line $\theta = 0$.

21. A wheel of radius r rolls along a straight line. Show that the distance moved per revolution by a point on the rim is $8r$.

CHAPTER XX

MEAN VALUE OF A FUNCTION APPROXIMATE INTEGRATION

111. Introduction.—The average, or arithmetic mean, of a set of n numbers is found by dividing their sum by n. Thus, the average of the numbers 6, 12, 14, 13, and 10, is

$$\frac{6 + 12 + 14 + 13 + 10}{5} = 11.$$

It is often desirable to assign different "weights" to the numbers, the weights indicating the relative importance of the numbers in the calculation. Thus, if one should drive for 7 hr. at 40 m.p.h., then for 2 hr. at 50 m.p.h. and then for 1 hr. at 60 m.p.h., his average speed for the 10 hr. would be

$$V_{av.} = \frac{40(7) + 50(2) + 60(1)}{7 + 2 + 1} = 44 \text{ m.p.h.}$$

This is not the arithmetic mean of the numbers 40, 50, and 60 but is a *weighted average*, the weights being the numbers 7, 2, and 1.

112. Mean value of a function.—Referring to Fig. 101 it is evident that the quantity

$$\frac{y_1\Delta x_1 + y_2\Delta x_2 + \cdots + y_n\Delta x_n}{\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n}$$

is a weighted average of the n ordinates shown in the interval from $x = a$ to $x = b$, the Δx's being the weights. The numerator in this expression is equal to the sum of the areas of the rectangles shown, while the denominator is equal to the length $b - a$ of the interval.

Suppose now that the number of ordinates selected in the interval is allowed to increase indefinitely in such a

way that each $\Delta x \to 0$. The numerator approaches the area under the curve while the denominator remains equal to $b - a$. It is natural then that we should define the *mean ordinate* to the curve, or the *mean value of the function*

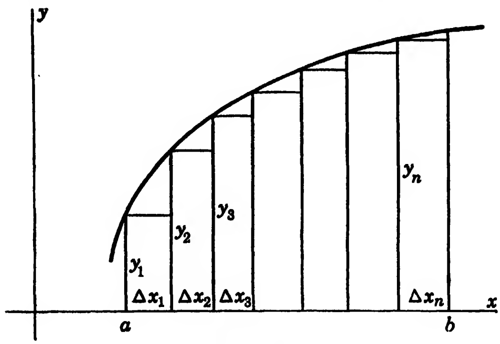

FIG. 101.

$f(x)$ in the interval from $x = a$ to $x = b$ by the equation

$$y_m = \frac{\int_a^b f(x)dx}{b - a}.$$

Example

The mean ordinate to the parabola $y = x^2$ in the interval from $x = 1$ to $x = 5$ is (Fig. 102)

$$y_m = \frac{\int_1^5 x^2 dx}{5 - 1} = 10\tfrac{1}{3}.$$

This is of course the height of a rectangle erected upon $b - a$ as base with area equal to the area under the curve.

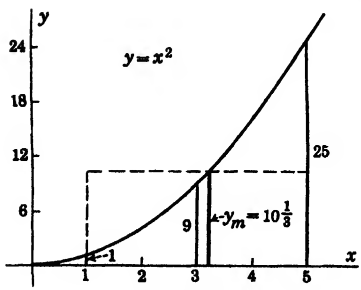

FIG. 102.

In many physical considerations the idea of the mean value of a function as above defined is important. A little reflection will convince the student that when he speaks of average velocity, average temperature, average force, etc., he is thinking of the mean value of a function in the sense here defined.

113. The polynomial $Ax^3 + Bx^2 + Cx + D$.—By direct integration one can show that *if $f(x)$ is any polynomial of degree 3 or less,* then

$$(1) \qquad \int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

It follows that the mean value of the polynomial in the interval may be found by adding its values at the ends of the interval, adding on four times its value at the mid-point of the interval, and dividing by 6. Thus the mean value of $f(x) = x^2$ in the interval from $x = 1$ to $x = 5$ is (Fig. 102)

$$y_m = \frac{1 + 4(9) + 25}{6} = 10\tfrac{1}{3}.$$

The relation (1) may of course be used to evaluate the definite integral of a polynomial of degree 3 or less, without performing the integration. Thus, since the values of $x^3 - 3x^2 + 5x + 1$ when $x = 0$, 2, and 4, are respectively 1, 7, and 37, we have

$$\int_0^4 (x^3 - 3x^2 + 5x + 1)dx = \frac{4-0}{6}[1 + 4(7) + 37] = 44.$$

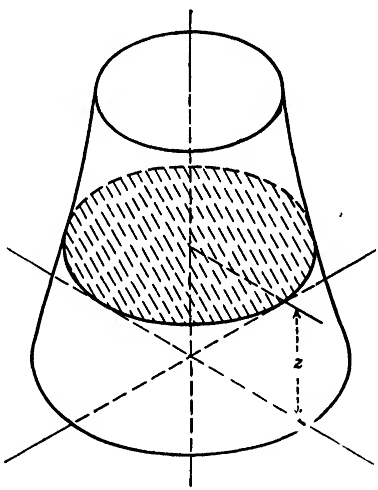

Fig. 103.

114. The prismoid formula. Suppose that at a distance z above the base, the cross-sectional area of the solid shown in Fig. 103 is $A(z)$. Its volume is then

$$V = \int_0^h A(z)dz.$$

If $A(z)$ is a polynomial of degree 3 or less, the value of this integral is, by (1),

$$V = \frac{h}{6}[A_B + 4A_M + A_T],$$

where A_B, A_T and A_M denote the cross-sectional areas at bottom, top, and mid-section, respectively. This *prismoid formula* gives exactly the volume of a solid satisfying the

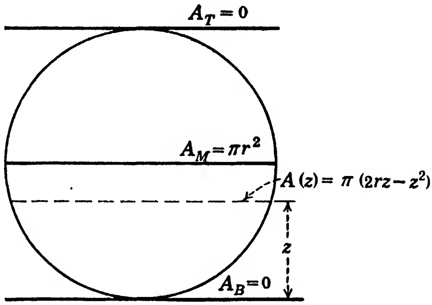

FIG. 104.

condition specified. Thus, for a sphere $A_T = A_B = 0$ and $A_M = \pi r^2$; hence (Fig. 104)

$$V = \frac{2r}{6}(0 + 4\pi r^2 + 0) = \frac{4}{3}\pi r^3.$$

The result is necessarily exact because at a distance z above the bottom section the cross-sectional area is $\pi(2rz - z^2)$, *a polynomial of degree* 2.

For solids not satisfying the specified conditions, the formula gives an approximation to the volume. It is used by engineers in estimating the volumes of such irregular solids as one encounters, for example, in making cuts and fills in road building.

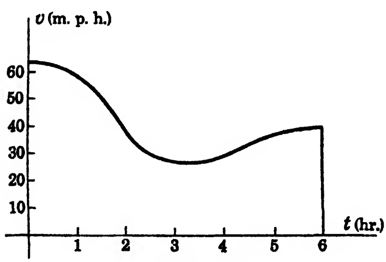

FIG. 105.

PROBLEMS

1. Suppose that one drives for 6 hr. with speed varying as indicated in Fig. 105. Show that the area under the curve represents the distance traveled and that the average speed would be obtained by dividing this area by the length of the time interval.

2. Sketch a curve which might represent the variation in temperature over a 24-hr. period. How would you obtain the average temperature?

3. Under what commonly occurring condition is the average value of a function $f(x)$ over an interval from $x = a$ to $x = b$ equal to one-half the sum of $f(a)$ and $f(b)$? Illustrate with a sketch.

4. Compute the mean ordinate to the curve $y = \cos x$ in the interval from $x = 0$ to $x = \frac{1}{2}\pi$.

5. Find the average value of $\sin^2 x$ from $x = 0$ to $x = \pi$.

6. What is the average value of $\sqrt{x}$ for values of x from 0 to 100?

7. What is the average length of the ordinates to the semicircle $y = \sqrt{r^2 - x^2}$?

8. Show that if $f(x) = Ax^3 + Bx^2 + Cx + D$, then,

$$\int_{-h}^{h} f(x)dx = \frac{2h}{6}[f(-h) + 4f(0) + f(h)].$$

Show that any interval, say $x = a$ to $x = b$, can be transformed into an interval from $x = -h$ to $x = +h$ by translation of the axes.

9. The points $P(1, 6)$ $Q(3, 8)$, $R(5, 2)$ determine a parabola of the form $y = Ax^2 + Bx + C$. Compute the area under it in the interval $x = 1$ to $x = 5$ without using its equation. Check by finding its equation and integrating.

10. An indefinite number of cubic curves of the form $y = Ax^3 + Bx^2 + Cx + D$ can be put through the points P, Q, and R of Prob. 9. What statement can be made about the areas under these curves in the interval from $x = 1$ to $x = 5$?

11. Compute the average ordinate to the curve $y = x^3 + 1$ from $x = -1$ to $x = 5$ in two different ways.

12. Evaluate $\int_0^2 (x^3 - 7x^2 + 2x - 6)dx$ by two methods.

13. Show that the prismoid formula gives exactly the volume of a right circular cone or cylinder.

14. Use the prismoid formula to find the volume common to two cylinders of radius r whose axes intersect at right angles.

15. An automobile headlight is in the form of a paraboloid of revolution. It is 8 in. in diameter and 6 in. deep. Show that the prismoid formula will give its volume exactly and compute the volume by this method.

16. A man drives at 40, 50, and 60 m.p.h. for 7, 2 and 1 hr., respectively. Compute his average speed, (*a*) with respect to time, and (*b*) with respect to distance. Explain why the results are different. Illustrate each with a graph.

115. Approximate integration—trapezoidal rule.—If the integration cannot be performed, the value of $\int_a^b f(x)dx$ can be approximated by plotting the curve whose equation

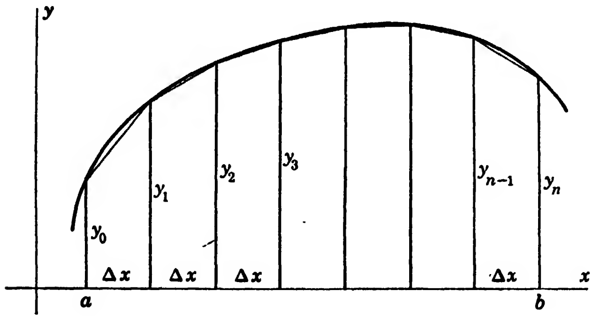

FIG. 106.

is $y = f(x)$ and approximating the area under it in the interval from $x = a$ to $x = b$.

One obvious method of making this approximation is to divide the interval from $x = a$ to $x = b$ into a convenient number n of equal parts, construct the trapezoids as indicated in Fig. 106, and compute the sum of their areas.

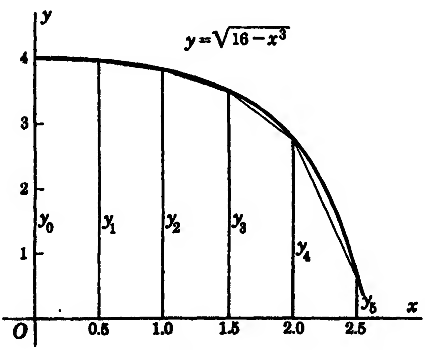

FIG. 107.

If we denote the $n + 1$ ordinates by $y_0, y_1, \cdots, y_n$, and the sum of the areas of the trapezoids by A_T we have

$$A_T = \tfrac{1}{2}(y_0 + y_1)\Delta x + \tfrac{1}{2}(y_1 + y_2)\Delta x + \cdots + \tfrac{1}{2}(y_{n-1} + y_n)\Delta x,$$

or,

$$A_T = \Delta x[\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n].$$

Example

Evaluate approximately $\int_0^{2.5} \sqrt{16 - x^3}\, dx$ taking $n = 5$.

Solution (Fig. 107)

Since $n = 5$, $\Delta x = \frac{1}{2}$. Computing the ordinates we have

x	0	0.50	1.00	1.50	2.00	2.50
y	4	3.98	3.87	3.55	2.83	0.61

$$A_T = \tfrac{1}{2}[\tfrac{1}{2}(4) + 3.98 + 3.87 + 3.55 + 2.83 + \tfrac{1}{2}(0.61)] = 8.27.$$

116. Simpson's rule.—Another method of approximating the area under the curve whose equation is $y = f(x)$, from $x = a$ to $x = b$, is as follows:

1. Divide the interval from $x = a$ to $x = b$ into an *even* number of equal parts. Denote the lengths of the $n + 1$ ordinates by $y_0, y_1, \cdots y_n$, as indicated in Fig. 108.

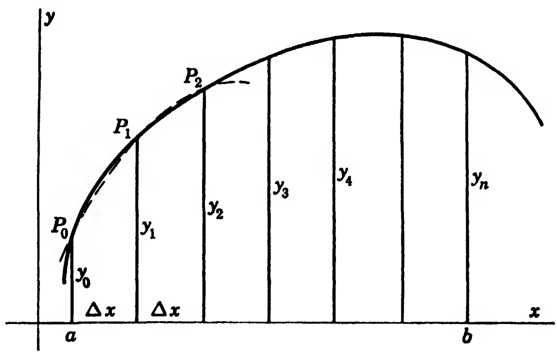

FIG. 108.

2. Take as an approximation to the area under the curve in the first *two* strips, the quantity

$$\frac{2\,\Delta x}{6}(y_0 + 4y_1 + y_2).$$

This is exactly the area under the parabola $y = Ax^2 + Bx + C$ determined by the points P_0, P_1, P_2, and shown dotted in the figure. The procedure then amounts to substituting an arc of a parabola for this part of the curve.

3. Similarly, approximate the area under the curve in the next two strips by $\frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$ and continue this process. Finally, adding together these approximations, we have

$$A_s = \frac{\Delta x}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)],$$

or,

$$A_s = \frac{\Delta x}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n].$$

From previous considerations it is clear that this formula will give the value of $\int_a^b f(x)dx$ *exactly*, if $f(x)$ is a polynomial of degree not higher than 3.

PROBLEMS

In each of the following, sketch the curve and evaluate the integral using both the trapezoidal rule and Simpson's rule. Check by performing the integration.

1. $\int_1^9 x^2dx;\ n = 8.$
2. $\int_4^7 \sqrt{x^2 + 9}\, dx;\ n = 6.$
3. $\int_0^3 x\sqrt{9 - x^2}\, dx;\ n = 6.$
4. $\int_1^4 \frac{dx}{x};\ n = 6.$
5. $\int_{-1}^5 (x + 1)(x - 5)(x - 7)dx;\ n = 6.$

Evaluate each of the following integrals by both methods if n is even, and by the trapezoidal rule only if n is odd:

6. $\int_0^4 \sqrt[3]{x^2 + 8}\, dx;\ n = 4.$
7. $\int_0^4 \sqrt{x^3 + 3x}\, dx;\ n = 4.$
8. $\int_0^5 \sqrt{125 - x^3}\, dx;\ n = 5.$
9. $\int_1^6 \frac{x^3dx}{\sqrt{x^3 + 2}};\ n = 5.$
10. $\int_0^{\frac{\pi}{2}} \sqrt{\sin^2 \theta + 4}\, d\theta;\ n = 6.$
11. $\int_{-1}^{+1} e^{-x^2}dx;\ n = 4.$
12. Evaluate $\int_0^4 (x^3 + 1)dx$ taking $n = 2$ and again taking $n = 4$ using Simpson's rule. Why are the results the same?

13. Compute approximately the area under the curve $y = \sqrt{\sin x}$ from $x = 0$ to $x = \frac{1}{2}\pi$. Take $n = 6$.

14. Compute approximately the length of one arch of the curve $y = \sin x$.

15. Compute approximately the length of the curve $y = x^3$ from $x = 0$ to $x = 4$.

CHAPTER XXI

FIRST MOMENT. CENTROID

117. First moment and centroid of the arc of a plane curve.—Consider the arc AB shown in Fig. 109 and suppose that we:

1. Divide the arc into n parts, $\Delta s_1, \Delta s_2, \cdots, \Delta s_n$.
2. Multiply the length of each piece by the distance of this particular piece from the x-axis; more precisely, multi-

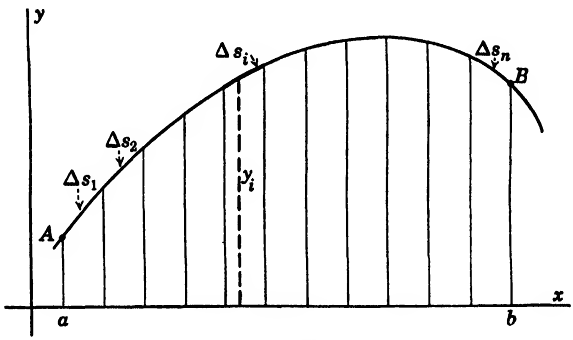

FIG. 109.

ply each Δs_i by the distance y_i from the x-axis to *any* point in this piece.

3. Add together the quantities so obtained; *i.e.*, form the sum

$$\sum_{i=1}^{n} y_i \Delta s_i.$$

4. Determine the limit of this sum as each $\Delta s_i \to 0$ and $n \to \infty$. This limit is called the *first moment of the length of arc* with respect to the x-axis and is denoted by M_x. Since each term in the sum differs only by an infinitesimal of higher order from $y_i ds_i$, the value of this limit is given by

$$M_x = \int_{x=a}^{x=b} y\,ds.$$

If the length of each piece of arc be expressed in inches, and its distance from the x-axis also in inches, the units of M_x are of course *inches squared.* Suppose now that we divide M_x by the total length of arc AB; the result, which is again in inches, may be regarded as an *average* distance of the arc from the x-axis. It is in fact the average *with respect to the arc* of the ordinates to the curve. Denoting it by $\bar{y}$ we have

$$\bar{y} = \frac{\int y\,ds}{s}.$$

In a similar manner we may define the corresponding average distance of the arc from the y-axis. Denoting it by $\bar{x}$, we have

$$\bar{x} = \frac{\int x\,ds}{s}.$$

The distances $\bar{x}$ and $\bar{y}$ determine a point C called the *centroid* of the arc. Physically, this point is the "center of gravity" of a thin wire bent in the form of the arc.

Example

Locate the centroid of the arc of a quadrant of the circle $x^2 + y^2 = r^2$.

Solution (Fig. 110)

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$$ds = \sqrt{1 + \frac{x^2}{y^2}}\,dx$$

$$= \frac{r}{y}dx.$$

$$M_x = \int_0^r y\left(\frac{r}{y}\right)dx$$

$$= \int_0^r r\,dx = r^2.$$

$$\bar{y} = \frac{M_x}{s} = \frac{r^2}{\frac{1}{2}\pi r} = \frac{2r}{\pi}.$$

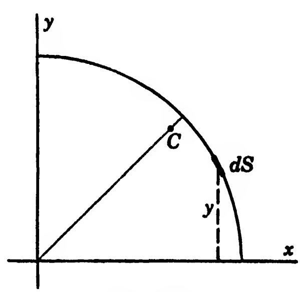

FIG. 110.

From the symmetry of the curve it is evident that $\bar{x} = \bar{y}$. The centroid of the arc is located at C in the figure.

PROBLEMS

1. Explain how the definition of $\bar{y}$ as given here is related to the idea of the mean ordinate to a curve as defined in the previous chapter. HINT: Consider carefully the quantities

$$\frac{\Sigma y_i \Delta x_i}{\Sigma \Delta x_i} \quad \text{and} \quad \frac{\Sigma y_i \Delta s_i}{\Sigma \Delta s_i}$$

noting that both are weighted averages of the ordinates.

2. Find $\bar{x}$ and $\bar{y}$ for the upper half of the arc of the circle $x^2 + y^2 = r^2$.

3. Solve Prob. 2 using polar coordinates.

4. A piece of wire having uniform cross-sectional area and density is bent into the form of a circular arc with central angle 60°. The radius is 6 in. Locate its center of gravity.

5. Find the centroid of the arc in the first quadrant of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

6. Locate the centroid of arc of half the hypocycloid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

7. Locate the centroid of arc of one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

8. Prove the First Theorem of Pappus: *If a curve which lies entirely on one side of a line is revolved about the line, the area of the surface generated is equal to the product of the length of the curve and the circumference of the path traveled by the centroid of the arc.* HINT: Revolve AB (Fig. 109) about the x-axis. Then $A = \int 2\pi y \, ds$ and $\int y \, ds = \bar{y}s$.

9. Use the Theorem of Pappus to compute the surface area of a torus.

10. Use the Theorem of Pappus to compute the lateral surface of a cone.

11. Using the Theorem of Pappus derive a formula for the lateral surface of a frustum of a cone.

12. Assuming the surface area of a sphere to be known, show how the Theorem of Pappus can be used to find the centroid of the arc of a semicircle.

118. First moment and centroid of a plane area.—Consider the plane area shown in Fig. 111 and suppose that we

1. Divide the area into a large number n of strips parallel to the x-axis and denote the areas of these strips by $\Delta A_1, \Delta A_2, \cdots, \Delta A_n$.

2. Multiply the area of each strip by the distance of this particular strip from the x-axis; more precisely, multiply each ΔA_i by the distance y_i from the x-axis to *any* point in this strip.

3. Add together the quantities so obtained; *i.e.*, form the sum

$$\sum_{i=1}^{n} y_i \Delta A_i.$$

4. Determine the limit of this sum as $\Delta A_i \to 0$ and $n \to \infty$. This limit is called the *first moment of the area* with

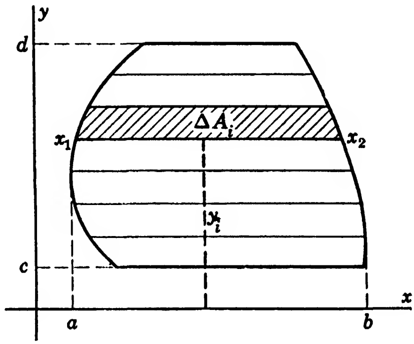

FIG. 111.

respect to the x-axis and is denoted by M_x. Since ΔA_i differs only by an infinitesimal of higher order from

$$dA_i = (x_2 - x_1)dy_i,$$

the value of this limit is given by

$$M_x = \int_{y=c}^{y=d} y\, dA$$

where dA can be expressed in terms of y and dy in an obvious way.

If the area of each strip be expressed in square inches, and its distance from the x-axis in inches, the units of M_x are of course *inches cubed.* Suppose now that we divide M_x by the entire area A; the result, which is again in inches, may be regarded as an average distance of the area from the x-axis. It is a weighted average of the distances of the strips from the x-axis, the weights being the areas of the strips. Denoting it by $\bar{y}$ we have

$$\bar{y} = \frac{\int y\, dA}{A}.$$

By dividing the area into strips parallel to the y-axis and proceeding as before, we may similarly define an average distance $\bar{x}$ of the area from the y-axis. The result is

$$\bar{x} = \frac{\int x\,dA}{A}.$$

The distances $\bar{x}$ and $\bar{y}$ determine a point C called the *centroid* of the area. Physically it represents the "center of gravity" of a thin plate whose shape is that of the area. The location of the centroid of an area is of importance in many types of engineering problems. It will be shown later, for example, that the resultant force exerted by a liquid against a plane submerged area can be found by multiplying the area by the pressure *at its centroid*. The role played by the centroid in connection with the computation of stresses in beams and columns is indicated in Art. 119.

Example 1

Locate the centroid of the area of a semicircle.

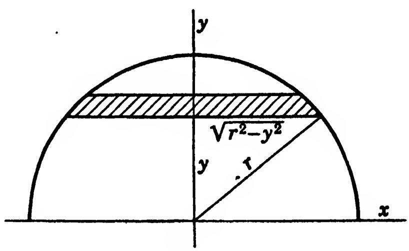

FIG. 112.

Solution (Fig. 112)

$$\begin{aligned}
dA &= 2\sqrt{r^2 - y^2}\,dy. \\
M_x &= \int_{y=0}^{y=r} y\,dA \\
&= \int_0^r 2y\sqrt{r^2 - y^2}\,dy \\
&= \frac{2r^3}{3}.
\end{aligned}$$

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{2r^3}{3}}{\frac{1}{2}\pi r^2} = \frac{4r}{3\pi}.$$

From the symmetry of the figure it is obvious that $\bar{x} = 0$.

Example 2

Show that the centroid of a rectangle is at its geometrical center.

Solution (Fig. 113)

$$dA = b\,dy$$

$$M_x = \int_0^h yb\,dy = \tfrac{1}{2}bh^2.$$

$$\bar{y} = \frac{\frac{1}{2}bh^2}{bh} = \tfrac{1}{2}h.$$

In the same way it is found that $\bar{x} = \frac{1}{2}b$.

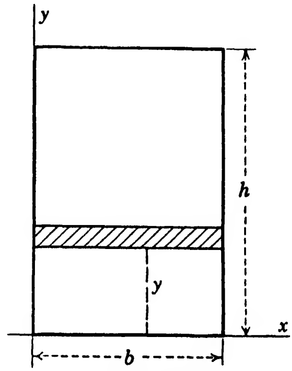

FIG. 113.

Sometimes, in order to obtain a simple integral, it is desirable to use strips *perpendicular* to the axis with respect to which the moment is to be found. With the result of the last example this can easily be done.

Example 3

Compute M_x for the area bounded by the x-axis and one arch of the curve $y = \sin x$.

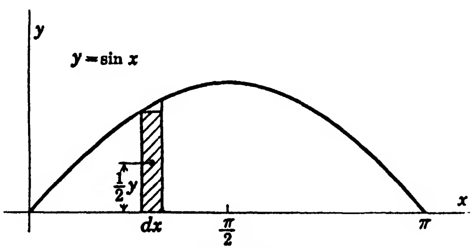

FIG. 114.

Solution (Fig. 114)

The area of the strip shown is $dA = y\,dx$. Since it is a rectangle, its moment with respect to the x-axis is

$$dM_x = (y\,dx)(\tfrac{1}{2}y) = \tfrac{1}{2}y^2dx.$$

We have, then,

$$M_x = \int_0^\pi \tfrac{1}{2}y^2 dx$$
$$= \tfrac{1}{2}\int_0^\pi \sin^2 x\, dx = \frac{\pi}{4}.$$

119. Composite areas.—Suppose that a beam, whose length is say 10 ft. and whose cross section has the shape shown in Fig. 115, is set up on simple supports with the 12-in. face horizontal and subjected to transverse loads which tend to bend it. It is fairly obvious that the longitudinal fibers in the lower part will be elongated and hence subjected to tensile stress, while those in the top part will be shortened and therefore subjected to compressive stress. In order to compute these stresses one must know the location of the *centroid* of the area. In fact it can be shown that, under certain conditions, the part *above* the centroid will be in compression and that *below* the centroid in tension; furthermore, the magnitude of the stresses at any distance y from the line through the centroid parallel to FG is directly proportional to y. Since the cross sections of beams and columns are often built up of simple rectangular and circular parts, it is necessary that the engineering student be able to locate the centroids of such composite areas.

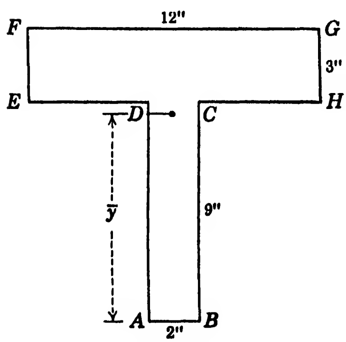

FIG. 115

Example

Find the centroid of the area shown in Fig. 115.

Solution

It is only necessary to compute the distance $\bar{y}$ of the centroid above AB.

Moment of rectangle $ABCD$ about AB = $(2)(9)(4\tfrac{1}{2})$ = 81 in^3.
Moment of rectangle $EFGH$ about AB = $(12)(3)(10\tfrac{1}{2})$ = 378 in^3.

$$\text{Total moment} = 81 + 378 = 459 \text{ in}^3.$$
$$\text{Total area} = 18 + 36 = 54 \text{ in}^2.$$
$$\bar{y} = \tfrac{459}{54} = 8\tfrac{1}{2} \text{ in.}$$

PROBLEMS

Find the centroids of the areas bounded as indicated:

1. $y^2 = 4x,\ x = 4.$ **2.** $y^2 = 8x,\ x^2 = 8y.$

3. $y = x^2 - 6x + 5,\ y = 0.$

4. $y = \cos x,\ x = 0,\ y = 0.$ First quadrant.

5. $y = \dfrac{rx}{h},\ y = 0,\ x = h.$ **6.** $y^2 = 10x,\ y = x.$

7. $\rho = a(1 + \cos\theta).$ **8.** $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}},\ x = 0,\ y = 0.$

9. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ x = 0,\ y = 0.$ First quadrant.

10. Find the centroid of the area of a triangle whose sides are 8, 8, and 4 in.

11. Find the centroid of the area under one arch of the cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta).$

12. Find the centroid of the area in the first quadrant bounded by the x-axis and the curve $y = 4x - x^3$.

13. Locate the centroid of the area shown in Fig. 116*a*.

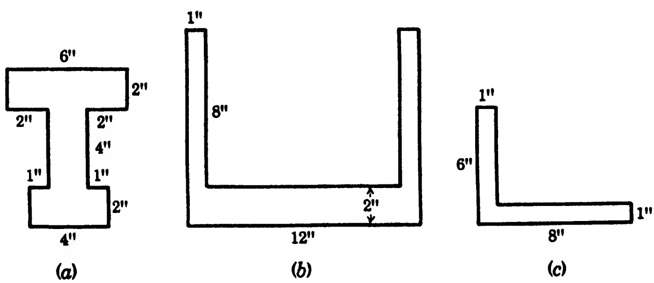

FIG. 116.

14. Locate the centroid of the area shown in Fig. 116*b*.

15. Locate the centroid of the area shown in Fig. 116*c*.

16. Prove the Second Theorem of Pappus: *If an area which lies entirely on one side of a line is revolved about the line, the volume of the solid generated is equal to the product of the area and the circumference of the path traveled by its centroid.*

17. Use the Second Theorem of Pappus to determine the volume of a torus.

18. Use the Second Theorem of Pappus to derive the formula for the volume of a cone.

19. Using the Second Theorem of Pappus derive a formula for the volume of a frustum of a cone, the radii of the bases being r and R and the height h.

20. Show how the Second Theorem of Pappus can be used to locate the centroid of the area of a semicircle, it being assumed that the formulas for area of a circle and volume of a sphere are known.

21. In what sense is the value of $\bar{y}$ for the upper half of the circle $x^2 + y^2 = r^2$ a weighted average of the ordinates? What are the weights?

120. First moment and centroid of volume and mass.—Consider the solid shown in Fig. 117 and suppose that we

1. Divide the solid into a large number n of slices by passing planes parallel to the xy-plane as indicated. Denote the masses of the slices by $\Delta M_1, \Delta M_2, \cdots, \Delta M_n$.

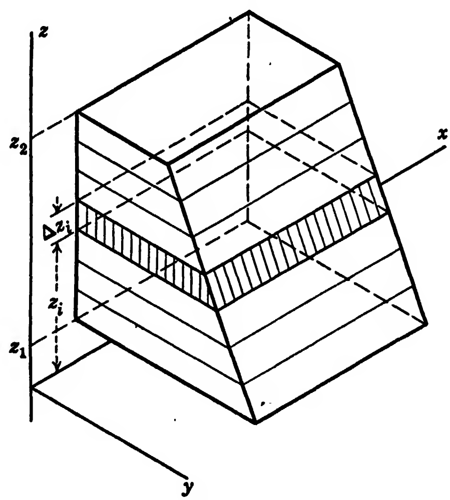

FIG. 117.

2. Multiply the mass of each slice by the distance of this slice from the xy-plane and add together the quantities so obtained; *i.e.*, form the sum

$$\sum_{i=1}^{n} z_i \Delta M_i.$$

3. Determine the limit of this sum as each $\Delta M \to 0$ and $n \to \infty$. This limit is called the *first moment of the mass* of the solid with respect to the xy-plane, and is denoted by M_{xy}. Its value is, of course, given by

$$M_{xy} = \int_{z=z_1}^{z=z_2} z\,dM, \qquad \text{where} \qquad dM = \delta A(z)dz,$$

δ being the mass per unit volume and $A(z)$ the area of a section parallel to the xy-plane and at a distance z from it.

Suppose now that we divide M_{xy} by the mass M of the solid. The result may be regarded as an average distance of the mass from the xy-plane; it is the z-coordinate of the centroid or center of gravity of the mass and is denoted by $\bar{z}$. We have then

$$\bar{z} = \frac{\int z\,dM}{M}.$$

In a similar manner we may define the other coordinates $\bar{x}$ and $\bar{y}$ of the centroid. The calculation can be carried out easily in simple cases in which the area of the cross section can be expressed as a simple function of the distance from the corresponding plane.

If the *volumes* of the slices were used instead of their masses, the *centroid of the volume* would be obtained. It is obvious that the centroid of the volume coincides with that of the mass if the solid is homogeneous.

Example

Find the centroid of the mass of a solid right circular cone of radius r and height h.

Solution (Fig. 118)

Taking the cone in the position shown we have $\bar{y} = \bar{z} = 0$. To find $\bar{x}$ we divide the solid into slices parallel to the yz-plane. Then, obviously,

$$dM = \delta\pi z^2 dx.$$

$$\int x\,dM = \delta\pi \int_0^h xz^2 dx;$$

but

$$\frac{z}{x} = \frac{r}{h}; \qquad \text{hence,} \qquad z^2 = \frac{r^2x^2}{h^2}.$$

Then

$$\int x\,dM = \frac{\delta\pi r^2}{h^2}\int_0^h x^3 dx$$
$$= \frac{\delta\pi r^2 h^2}{4}.$$

Dividing this by the mass of the cone ($= \frac{1}{3}\pi r^2 h\delta$), we have

$$\bar{x} = \tfrac{3}{4}h.$$

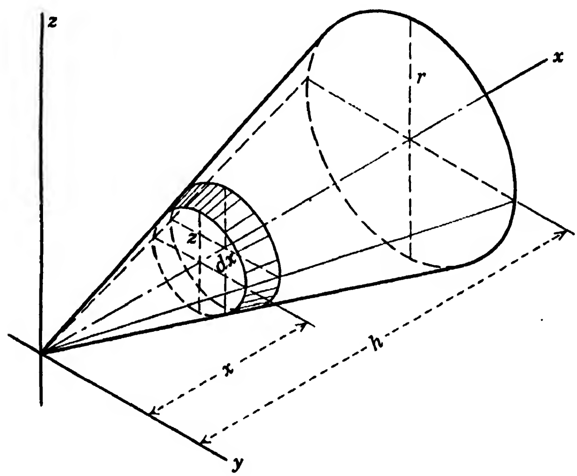

FIG. 118.

PROBLEMS

1. Find the centroid of a solid rectangular parallelepiped with edges a, b, and c.

2. Locate the centroid of a pyramid whose base is a square 4 ft. on a side if the vertex is 12 ft. above the center of the base.

3. A pyramid has a square base with each side 8 in. The height is 16 in., the vertex being vertically above one corner of the base. Find the centroid of its volume.

4. Find the centroid of the volume bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

5. Find the centroid of the volume of a right circular cone.

6. The density of a solid right circular cylinder of radius r and height h varies directly as the distance from one base. Find the centroid of its mass.

7. The density of a solid right circular cone varies directly as the distance from its base. Find its center of gravity.

8. Find the centroid of the volume of a hemisphere.

9. Locate the centroid of the mass of a solid hemisphere if the density varies directly as the distance from its plane surface.

10. Find the centroid of the volume bounded by the paraboloid $y^2 + z^2 = 6x$ and the plane $x = 6$.

11. Find the centroid of the volume bounded by the plane $z = 12$ and the surface $z = 16 - x^2 - 4y^2$.

12. The base of a right cone whose altitude is 24 in. is an ellipse with axes 16 and 9 in. Locate the centroid of its volume.

13. A solid cylindrical steel rod is tapered to a conical point, as indicated in Fig. 119. Locate its centroid.

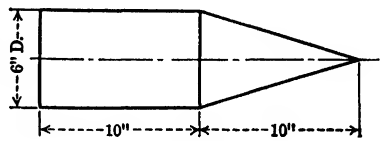

FIG. 119.

14. A water tank consists of a right circular cylinder with its axis vertical, to which is attached a hemispherical bottom. The diameter and height of the cylinder are each 12 ft. Find the center of gravity of the water when the tank is full.

15. Find the centroid of the half of a torus generated by revolving the upper half of the circle $(x - R)^2 + y^2 = r^2$ about the y-axis.

16. Find the centroid of the volume generated by revolving a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.

17. Find the centroid of the volume of a conoid of height h, the base being the circle $x^2 + y^2 = r^2$.

18. A solid right circular cylinder of radius r and height h is cut by a plane which passes through a diameter of one base and is tangent to the other base. Find the centroid of the smaller piece.

CHAPTER XXII

SECOND MOMENT. RADIUS OF GYRATION

121. Second moment of area.—The definition of the *second moment* or *moment of inertia* of a plane area with respect to a line differs from that of the first moment only in one respect; namely, the area of each strip is multiplied by the *square* of the distance from the line instead of by the first power of the distance. The second moments with respect to the x-axis and y-axis are denoted by I_x and I_y, respectively. We have, then,

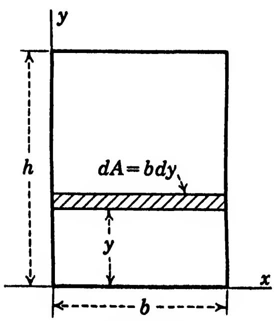

FIG. 120.

$$I_x = \int y^2 dA, \qquad I_y = \int x^2 dA,$$

where dA denotes in each case the area of a strip parallel to the proper axis.

Example 1

Compute the second moment of the area of a rectangle with respect to one side.

Solution (Fig. 120)

$$dA = b\,dy$$
$$y^2 dA = y^2 b\,dy$$
$$I_x = \int_0^h y^2 b\,dy = \frac{bh^3}{3}.$$

Example 2

Compute I_y for the area of the circle shown in Fig. 121.

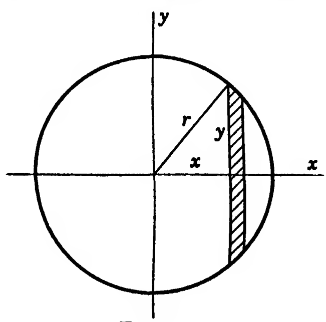

FIG. 121.

Solution

$$dA = 2y\,dx$$
$$x^2 dA = 2x^2 y\,dx = 2x^2\sqrt{r^2 - x^2}\,dx;$$
$$I_y = \int_{-r}^{r} 2x^2\sqrt{r^2 - x^2}\,dx = \frac{\pi r^4}{4}.$$

Could this result have been obtained by finding I_y for the area in the first quadrant and multiplying by 4? Explain.

122. The Transfer Theorem.—When the value of I with respect to a line through the centroid of an area is known, that with respect to any line parallel to this can be found easily using the following:

Theorem: *If I_g is the second moment of an area A with respect to a line through its centroid, and I_L is the second moment with respect to any line parallel to this, then*

$$I_L = I_g + Ad^2$$

where d is the distance between the two lines.

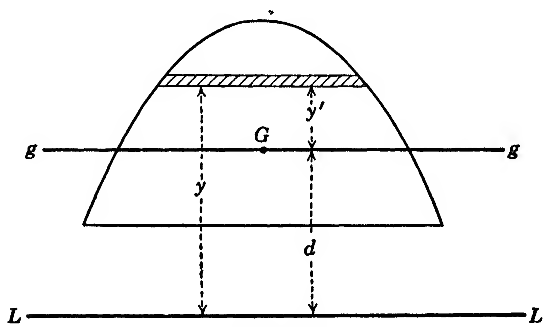

FIG. 122.

PROOF: Using Fig. 122, we have

$$\begin{aligned} I_L &= \int y^2 dA \\ &= \int (y' + d)^2 dA \\ &= \int y'^2 dA + 2d\int y' dA + d^2 \int dA \\ &= I_g + 0 + Ad^2. \end{aligned}$$

The student should be careful to note that the relation is not true for *any* two parallel lines; *i.e.*, $\int y' dA = 0$ only if this line passes through the centroid.

Example

Compute the value of I for the area of a circle with respect to a tangent line.

Solution (Fig. 123)

We have already found that, with respect to a diameter,

$$I_x = \frac{\pi r^4}{4}.$$

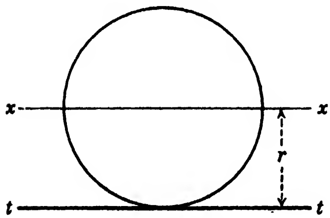

FIG. 123.

Applying the Transfer Theorem we have

$$I_t = \frac{\pi r^4}{4} + \pi r^2 \cdot r^2$$

$$= \frac{5\pi r^4}{4}.$$

123. Composite areas.—Of great importance in connection with the computation of stresses in beams and columns, is the second moment of the area of the cross section of the beam or column with respect to a line called the *neutral axis* of the section. The section often consists of rectangles, and the value of I for a rectangle with respect to a line through its centroid (parallel to side b) is (Fig. 124)

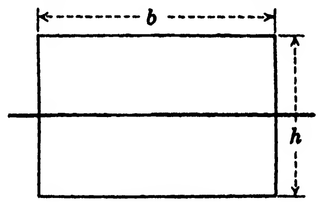

FIG. 124.

$$I = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b\, dy = \frac{bh^3}{12}.$$

One may use this result and the transfer theorem to compute I for a composite area.

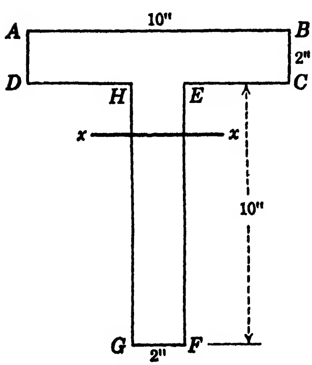

FIG. 125.

Example

For the area shown in Fig. 125, compute the value of I with respect to a line through the centroid parallel to AB.

Solution

To find the distance from AB to the centroid, we have

$$\bar{y} = \frac{20(1) + 20(7)}{40} = 4 \text{ in.}$$

For $ABCD$ the value of I_x is

$$I_1 = \frac{10 \cdot 2^3}{12} + 20(3)^2 = 186.7 \text{ in}^4.$$

For $EFGH$ the value of I_x is

$$I_2 = \frac{2 \cdot 10^3}{12} + 20(3)^2 = 346.7 \text{ in}^4.$$

Adding these we have for the entire area

$$I_x = 186.7 + 346.7 = 533.4 \text{ in}^4.$$

124. Radius of gyration.—If all of the dimensions involved are expressed in inches, the value of I for an area with respect to a line is of course expressed in *inches to the fourth power;* this is true because it is obtained by multiplying the area of each strip by the *square* of a distance and adding these quantities. Suppose now that we divide the value of I by the total area A. The result, which is in *inches squared,* may be regarded as an *average of the squares* of the distances of the strips from the line. The square root of this average, which is of course in inches, is called the *radius of gyration* of the area with respect to the line. Denoting it by k we have

$$k = \sqrt{\frac{I}{A}}.$$

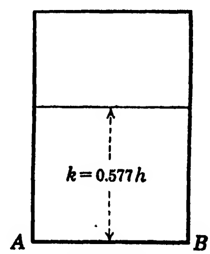

FIG. 126.

Example

Find the radius of gyration of the area of the rectangle shown in Fig. 126 with respect to the base AB.

Solution

We have already found that, with respect to AB,

$$I = \frac{bh^3}{3}.$$

Dividing this by the area and taking the square root we have

$$k^2 = \frac{h^2}{3}$$

$$k = \frac{h}{\sqrt{3}} = 0.577h.$$

PROBLEMS

Compute I_x and I_y for the area bounded as indicated:

1. The area bounded by $y^2 = 4x$ and $x = 4$.

2. The area bounded by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

3. The area in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$.

4. The area bounded by $y = x^3$, $y = 0$, and $x = 2$.

5. Find I_x for the area under one arch of the curve $y = \sin x$ using strips *perpendicular* to the x-axis. HINT: The value of I_x for one of the rectangular strips is $y^3dx/3$. Why?

6. Compute I_y for the area bounded by the x-axis and the parabola $y = 4x - x^2$. Find the corresponding radius of gyration.

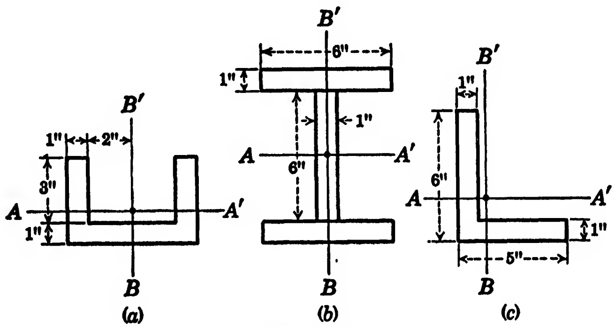

FIG. 127.

7. Compute I_y for the area bounded by the coordinate axes, the curve $y = \dfrac{8}{x^2 + 4}$, and the line $x = 2$.

8. Compute I_x and I_y for the area bounded by the coordinate axes, the curve $y = e^x$, and the line $x = 1$.

9. Show that for a right triangle with vertices at (0, 0), $(b, 0)$, and $(0, h)$, the value of I_x is $bh^3/12$.

10. Show that for any triangle with base b and altitude h, $I_b = bh^3/12$.

11. Find the value of I for the area of a triangle of base b and altitude h with respect to a line through the centroid parallel to the base. HINT: Compute I with respect to base (Prob. 10) and use the Transfer Theorem.

12. Compute the value of I for the area of a square with respect to a diagonal.

13. Find the value of I for the area of a circle with respect to a tangent line both with and without using the Transfer Theorem.

For each of the following areas compute the values of I with respect to lines AA' and BB' which go through the centroid:

14. The area of Fig. 127*a*.

15. The area of Fig. 127*b*.

16. The area of Fig. 127*c*.

17. An isosceles trapezoid has bases 12 and 8 in. and altitude 8 in. Find the value of I with respect to the longer base.

18. Suppose that an area is symmetrical with respect to the x-axis. What general statement can be made about the values of the first and second moments of the area with respect to this axis?

19. The first moment of the area of the circle $x^2 + y^2 = r^2$ with respect to the x-axis is *zero*, that of the bottom half canceling that of the top half. Why is this not true also of the second moment?

20. The area of the cross section of a 12-in. American Standard I-beam is 14.57 sq. in. The radii of gyration with respect to its two axes of symmetry are, respectively, 4.55 and 1.05 in. Compute the corresponding values of I.

125. Second moment of mass with respect to a plane.—The second moment of the mass of a body with respect to a

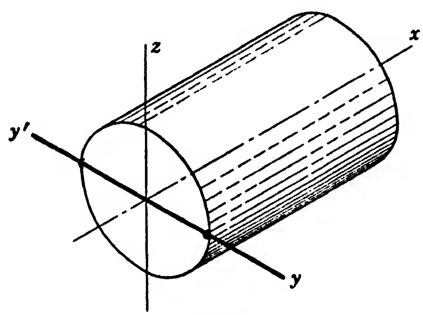

FIG. 128.

plane is defined in the same way as the first moment, the only difference being that the mass of each slice is multiplied by the *square* of its distance from the plane instead of by the first power of this distance. This quantity is not itself of physical importance. It is used, however, in finding the second moment of a mass with respect to a line—and this latter quantity is of fundamental importance in the dynamics of rotating bodies.

Suppose, for example, that a solid cylinder is held with its axis horizontal as indicated in Fig. 128 and then released to swing as a pendulum about the fixed axis yy'. Let it be required to determine the rotational velocity of the cylinder

when it passes through the lowest position, and the pull which it exerts upon the supports at that instant. These are problems of mechanics which we shall not solve here. We wish merely to point out that in order to solve them one must know the value of a quantity called the *second moment* or *moment of inertia* of the mass of the rotating body with respect to the axis of rotation. This quantity is defined and discussed in the next section.

126. Second moment of mass with respect to a line.— Think of the solid cylinder shown in Fig. 128 as being composed of millions of little particles. Suppose that we *multiply the mass of each particle by the square of its distance from the axis* yy', and add together the quantities thus obtained. The resulting quantity is called the *second moment* or *moment of inertia* of the mass of the body with respect to the line yy'. More precisely, the second moment is the *limit* approached by this sum when the number of particles is increased indefinitely in such a way that the largest dimension of each particle approaches zero. We denote it by I_y.

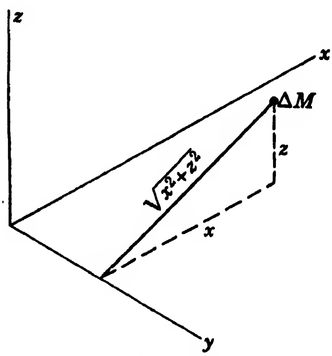

FIG. 129.

A method of calculating this quantity results from the simple observation, made clear by Fig. 129, that for *any* particle of mass ΔM,

$$I_y = (x^2 + z^2)\Delta M = x^2\Delta M + z^2\Delta M = I_{yz} + I_{xy}.$$

That is, for each particle, the value of I with respect to the line yy' is equal to the sum of the values of I with respect to the two planes that intersect in this line. Since this is true for each particle, it is true for the sum of n particles and for the limit of the sum. We have, then, the following:

Theorem: *The second moment of the mass of a body with respect to a line is equal to the sum of its second moments with*

respect to any two planes which intersect at right angles along this line.

In the case of the cylinder shown in Fig. 128 the value of I_{yz} can be found easily by dividing it into slices parallel

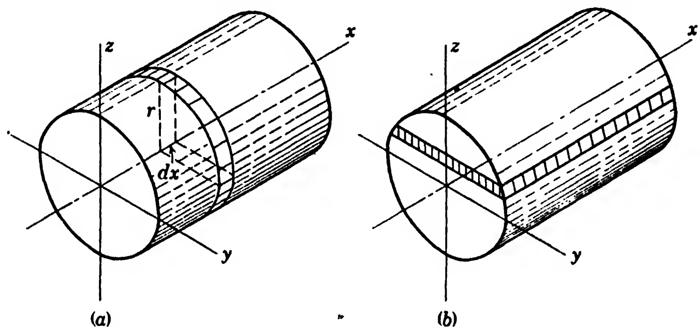

FIG. 130.

to the yz-plane. Thus (Fig. 130a),

$$I_{yz} = \int x^2 dM = \int_0^h x^2 \cdot \delta\pi r^2 dx = \frac{\delta\pi r^2 h^3}{3} = \tfrac{1}{3}Mh^2.$$

Similarly, using slices parallel to the xy-plane (Fig. 130b),

$$\begin{aligned} I_{xy} = \int z^2 dM &= \int_{-r}^{r} z^2 \cdot \delta 2\sqrt{r^2 - z^2}\, h\, dz \\ &= 4\delta h \int_0^r z^2 \sqrt{r^2 - z^2}\, dz \\ &= 4\delta h r^4 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta \text{ (letting } z = r\sin\theta) \\ &= \frac{\delta\pi r^4 h}{4} = \tfrac{1}{4}Mr^2. \end{aligned}$$

Adding the results, we have

$$I_y = \tfrac{1}{3}Mh^2 + \tfrac{1}{4}Mr^2.$$

127. Solids of revolution.—For a solid of revolution, the value of I with respect to the axis of revolution can be found by the cylindrical shell method. Thus, for a solid cylinder (Fig. 131), we have

$$I_x = \int_0^r y^2 \cdot \delta 2\pi y h\, dy = \frac{\delta\pi h r^4}{2} = \tfrac{1}{2}Mr^2.$$

Using this result one can, for a solid of revolution, calculate the value of I with respect to the axis of revolution by dividing it into disks. Thus, for a solid cone we have (Fig. 132): For slice shown

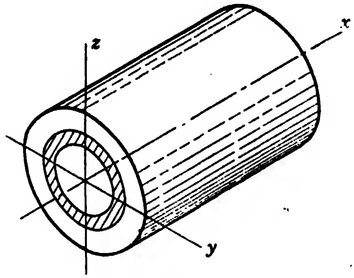

FIG. 131.

$$I_z = \tfrac{1}{2}(\Delta M)x^2 = \tfrac{1}{2}\pi\delta x^4 dz.$$

For the whole cone,

$$I_z = \tfrac{1}{2}\pi\delta\int_0^h x^4 dz.$$

And, since

$$\frac{x}{r} = \frac{h-z}{h},$$

$$I_z = \frac{\delta\pi r^4}{2h^4}\int_0^h (h-z)^4 dz$$

$$= \frac{\pi\delta r^4}{2h^4}\cdot\frac{h^5}{5} = \tfrac{3}{10}Mr^2.$$

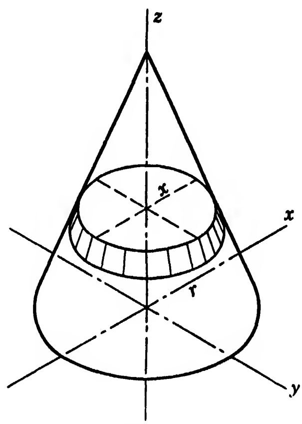

FIG. 132.

128. The Transfer Theorem. *Suppose that g is a line which passes through the centroid of a solid whose mass is M; suppose that L is a line parallel to g and that the distance between L and g is d. Then*

$$I_L = I_g + Md^2.$$

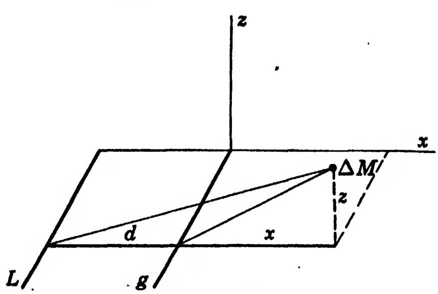

FIG. 133.

PROOF: Take the plane determined by L and g as the xy-plane and let the y-axis coincide with g as indicated in

Fig. 133. Then for *any* particle of mass ΔM in the body

$$\begin{aligned} I_L &= [z^2 + (x + d)^2]\Delta M \\ &= (z^2 + x^2)\Delta M + (2d)x\Delta M + d^2\Delta M. \end{aligned}$$

Summing these three terms separately over the whole body it is evident that the quantities obtained are respectively I_G, 0, and Md^2. Hence,

$$I_L = I_G + Md^2.$$

129. Radius of gyration.—Suppose that the value of I for the mass of a body with respect to a line be divided by the mass M of the body. The result may be regarded as a weighted average of the squares of the distances of the particles from the line. The square root of this average is called the *radius of gyration* of the mass with respect to the line. It represents the distance from the line at which *all* of the mass could be considered concentrated without changing its value of I. Denoting it by k, we have

$$k = \sqrt{\frac{I}{M}}.$$

Consider, as an example, the case of the solid cylinder. We have already found that, with respect to its axis, $I = \frac{1}{2}Mr^2$. Dividing by M and taking the square root, we have

$$k = \frac{r}{\sqrt{2}} = 0.707r.$$

That is, the value of I for this mass is the same as if all of the mass were concentrated at a distance of about $0.7r$ from the axis.

The moment of inertia with respect to the axis of rotation of an irregular rotating part such as a crankshaft, flywheel, or connecting rod, is often specified most conveniently by stating its radius of gyration or "equivalent radius." The value of I is computed from this by means

of the relation

$$I = Mk^2.$$

There are experimental methods of determining I or k for such irregular bodies. They involve such procedures, for example, as using the body as a compound pendulum or torsion pendulum, the value of I or k being computed from the observed periods of the motion.

130. Polar second moment of area.—Corresponding to the second moment of the mass of a cylinder with respect to its axis we have, in two dimensions, the second moment of the area of a circle with respect to its center. It is often called the *polar moment of inertia* of the area. It can be calculated by dividing the circle into concentric rings, multiplying the area of each ring by the square of its distance from the center, and adding. We have, then (Fig. 134),

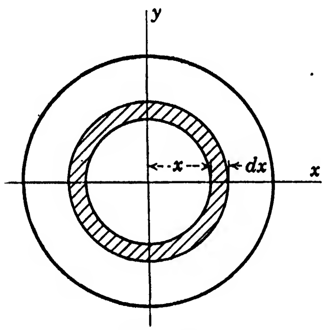

FIG. 134.

$$I_0 = \int_0^r x^2(2\pi x\, dx) = \frac{\pi r^4}{2}.$$

This quantity plays an important role in the calculation of the maximum shearing stress produced in a circular shaft when it is subjected to a twisting moment—as is the case when it is transmitting power.

One can define, in an obvious way, the second moment of any area with respect to a point. It can be shown that its value is equal to the sum of the second moments of the area with respect to any two lines which intersect at right angles at this point.

PROBLEMS

Compute the value of I for each of the following solids with respect to the line indicated:

1. A right circular cone with respect to its axis, using the cylindrical shell method.

2. A sphere with respect to a diameter. Use two methods.

3. A right circular cylinder with respect to an element.

4. A cube with respect to one edge.

5. A long slender rod with respect to an axis through one end perpendicular to the rod.

6. A right pyramid with square base of side a and altitude h, with respect to its axis.

7. A right circular cone with respect to a line through the apex perpendicular to the axis.

8. The ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{b^2} = 1$ with respect to the x-axis.

9. Compute I_x for the solid generated by revolving the area bounded by $y^2 = 4x$ and $x = 4$ about the x-axis. Use two methods.

10. Compute I_x for the solid generated by revolving one arch of the sine curve about the x-axis. HINT: Divide it into disks.

11. Find I_z for the volume of the torus generated by revolving the area of the circle $(x - R)^2 + z^2 = r^2$ about the z-axis. HINT: Use cylindrical shells.

12. Show that one may compute the value of I for a solid right circular cone with respect to a diameter of the base as follows: Add the value of I with respect to the plane of the base to one-half the value of I with respect to the axis of the cone.

13. We may *define* the second moment of the mass of a sphere *with respect to its center* by the integral $I_0 = \int \rho^2\, dM$ where ρ represents the distance from the center of the sphere to the particle. Show how to evaluate I_0 using concentric spherical shells. Show that the value of I with respect to a diameter is equal to two-thirds of this quantity.

14. A solid sphere rotates about an axis which is tangent to its surface. Compute the value of I and the corresponding radius of gyration with respect to this axis.

15. A pendulum consists of a long slender rod of length L and mass M_1, to the lower end of which is attached a solid sphere of radius r and mass M_2. Write down an expression for its moment of inertia with respect to the axis of rotation using the results of Prob. 2 and 5 and the Transfer Theorem.

16. A flywheel consists of a hollow cylindrical hub, several cylindrical spokes, and a heavy rim with square cross section. Explain how one could compute its moment of inertia with respect to the axis of revolution.

17. Compare the values of I (with respect to their geometrical axes) of a solid disk and a hoop having the same mass and radius.

18. A flywheel which is 8 ft. in diameter weighs 2 tons, the weight being distributed in such a way that its radius of gyration is 3.2 ft. Compute the value of I. HINT: Note that $I = Mk^2$ where $M = W/g$.

19. Find the second moment of the area of a square with respect to one corner.

20. Find the second moment of the area of a rectangle with respect to the center.

21. A hollow steel shaft has internal diameter 10 in. and external diameter 12 in. Find the polar second moment of the area of its cross section.

22. A hollow steel shaft with outside radius r has its inside radius equal to $\frac{1}{2}r$. Show that the polar moment of the area of its cross section is $\frac{15}{16}$ of that of a solid shaft of radius r.

CHAPTER XXIII

LIQUID PRESSURE. WORK

131. Fluid pressure.—Using the equilibrium principle of statics, it can be shown that the pressure at any point in a liquid is the same in all directions. It can furthermore be demonstrated that at a depth of h ft. below the free surface of a liquid which weighs w lb. per cubic foot, the pressure is

$$p = wh \text{ lb. per square foot.}$$

The pressure (per unit area) thus increases *linearly* with increasing depth, the rate of increase being equal to the weight per cubic unit of the liquid. In this chapter we shall always use the term pressure to mean the *force per unit area.*

132. Resultant force on a plane vertical wall.—Assuming the above principle, we may calculate the magnitude of the total force or *resultant force* against a plane submerged surface as indicated in the following example.

Example

Compute the resultant force exerted by the water against the face of the dam shown in Fig. 135 when the water is 12 ft. deep.

Solution

1. Divide the submerged area into n strips parallel to the surface of the water.

2. The force against each strip can be obtained, except for an infinitesimal of higher order, by multiplying the area of the strip by the pressure of *any* point in the strip; *i.e.*, except for an infinitesimal of higher order,

$$\begin{aligned} \Delta F_i &= wh_i(\Delta A_i) \\ &= w(12 - y_i)(20\,\Delta y_i). \end{aligned}$$

3. The sum of these quantities is a good approximation to the total force if each Δy is small, and the limit of the sum as each $\Delta y \to 0$ is exactly the required force; *i.e.*,

$$\begin{aligned} F &= \int_0^{12} w(12 - y)20\, dy \\ &= 1{,}440w = 1{,}440(62.4) = 89{,}856 \text{ lb.} \end{aligned}$$

133. Center of pressure.—The forces ΔF_1, ΔF_2, $\cdots$, ΔF_n, of the water against the strips of area in Fig. 135 constitute a system of parallel forces. To determine completely the resultant of such a system we must know not

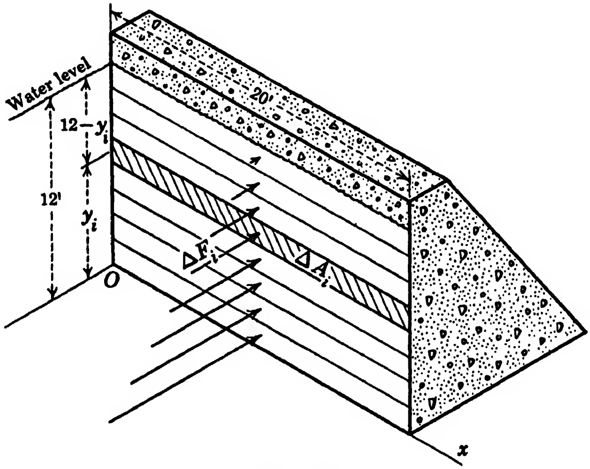

FIG. 135.

only its magnitude, which we have already found, but also its position; *i.e.*, we must find the height y' above the ground at which the resultant force of 89,856 lb. must be assumed to act. This may be found as follows:

1. Multiply the magnitude of each force ΔF_i by its distance above the bottom of the dam. This gives the *moment* of the force about the line Ox.

2. Add together these moments and find the limit of their sum as $\Delta y_i \to 0$. This gives the total moment of the force system. The result is

$$\begin{aligned} M &= \int_0^{12} yw(12 - y)(20\, dy) \\ &= 5{,}760w \text{ ft.-lb.} \end{aligned}$$

3. The resultant force of 1,440w lb. must be assumed to act at a distance y' above the ground such that its moment with respect to Ox is equal to the value just obtained; *i.e.*,

$$1{,}440wy' = 5{,}760w$$

or

$$y' = 4 \text{ ft.}$$

It is clear that the water tends both to push the dam back and to turn it over. We have found that, with respect to both of these items, the effect is the same as that of a concentrated force of 89,856 lb. applied at a point 4 ft. above the base. The point of application of this resultant force is called the **center of pressure.**

134. The general case of a submerged plane surface.—Consider a submerged thin plate whose plane makes an

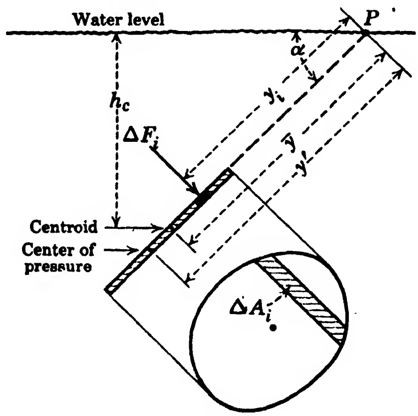

FIG. 136.

angle α with the plane of the liquid surface and intersects this surface in a line PP'. In Fig. 136 this line is perpendicular to the paper and is represented by point P.

The resultant force of the liquid against one side of the plate is normal to this surface and we state now two general theorems concerning its magnitude and position.

Theorem I: *The magnitude of the resultant force is equal to the product of the area and the pressure at its centroid,*

That is

$$F = wh_cA$$

where h_c *denotes the depth of the centroid below the free surface of the liquid.*

Theorem II: *The distance from the line PP′ to the center of pressure, measured along the plane of the plate, is*

$$y' = \frac{k^2}{\bar{y}},$$

where k *is the radius of gyration of the area with respect to the line PP′ and* $\bar{y}$ *is the distance from PP′ to the centroid.*

The proofs of these theorems are given below, some details being left to the student.

PROOF OF THEOREM I: Dividing the area into strips and noting that the depth of ith strip is $y_i \sin \alpha$ we have, for the force on this strip,

$$\Delta F_i = w(y_i \sin \alpha)\Delta A_i.$$

The total force on the plate is, then,

$$\begin{aligned} F &= \int wy \sin \alpha \, dA \\ &= w \sin \alpha \int y \, dA. \end{aligned}$$

The value of $\int y \, dA$ is the first moment of the area of the plate with respect to PP' and is of course equal to $\bar{y}A$ where $\bar{y}$ is measured from PP' as indicated in the figure. Hence,

$$F = w \sin \alpha \, \bar{y}A.$$

But $\bar{y} \sin \alpha$ is equal to the vertical distance h_c from the liquid surface to the centroid; consequently,

$$F = wh_cA.$$

PROOF OF THEOREM II: Multiplying the force against the ith strip by y_i we obtain the moment of this force with respect to line PP'. Thus,

$$\Delta M_i = y_i(wy_i \sin \alpha \, \Delta A_i).$$

The total moment of the force system with respect to PP' is, then,

$$M = w \sin \alpha \int y^2 dA.$$

But $\int y^2 dA$ is the second moment of the area of the plate with respect to PP' and is equal to Ak^2 where k is the corresponding radius of gyration. Consequently,

$$M = w \sin \alpha \, Ak^2.$$

The resultant force, whose magnitude was obtained in theorem I, must act at a distance y' from PP' so that its moment with respect to PP' is equal to the value just obtained; *i.e.*,

$$(w \sin \alpha \, \bar{y} A) y' = w \sin \alpha \, Ak^2.$$

Solving for y' we have

$$y' = \frac{k^2}{\bar{y}}.$$

Example

Find the magnitude and position of the resultant force against the dam shown in Fig. 135, using the above general theorems.

Solution

The depth of the centroid is 6 ft. and the pressure at this point is $6w$ lb. per square foot. The area is 240 sq. ft. The resultant force is, then,

$$F = 240(6w) = 1{,}440w \text{ lb.}$$

The value of k^2, with respect to the water line, is

$$k^2 = \frac{I}{A} = \frac{\frac{20(12^3)}{3}}{240} = 48.$$

The distance of the center of pressure below the surface of the water is

$$y' = \frac{k^2}{\bar{y}} = \frac{48}{6} = 8 \text{ ft.}$$

PROBLEMS

1. If y is the depth of the upper edge of a strip whose area is ΔA, show that the force on the strip differs from $wy \, \Delta A$ only by an infinitesimal of higher order. HINT: Since the pressure increases with increasing

depth, the force must be more than $wy\,\Delta A$ but less than $w(y + \Delta y)\Delta A$. Show that the difference between these extremes is of higher order than $wy\,\Delta A$.

2. Does the force exerted by the water against a dam depend upon the amount of water in the reservoir? Explain.

3. A rectangular plate is 3 ft. long and 2 ft. wide. It is submerged vertically in water with the upper 3 ft. edge parallel to and 4 ft. below the surface. Find the magnitude and position of the resultant force against one side of the plate both with and without using theorems I and II.

4. The edges of a triangular plate are 5, 5, and 6 ft. It is submerged vertically in water with the 6 ft. edge in the water surface. Find the force against one side both with and without using theorem I.

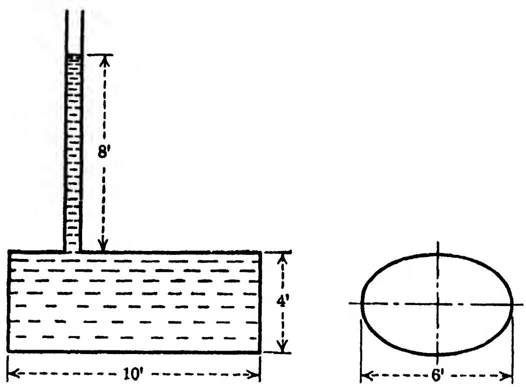

FIG. 137.

5. The edges of a cubical box are each 6 ft. The box contains liquid concrete weighing 250 lb. per cubic foot to a depth of 3 ft. Find the magnitude and position of the resultant force against one side.

6. Suppose that in Prob. 5 the side is held in place by four screws, one at each corner. Compute the tensile forces in these screws.

7. A cylindrical tank 4 ft. in diameter and 6 ft. long has its axis horizontal. It is half full of oil weighing 50 lb. per cubic foot. Find the force exerted against one end both with and without using theorem I.

8. A circular gate in the vertical face of a dam is 4 ft. in diameter. Find the force on it when the water level is 3 ft. above the top of the gate.

9. The cross section of an oil tank is an ellipse with major and minor axes 6 and 4 ft., respectively. The tank has its axis and the major axis of the ends horizontal. Find the force on the upper half and on the lower half of one end when it is full of oil weighing 50 lb. per cubic foot.

10. Find the force against one end of the elliptical tank shown in Fig. 137.

11. The two sloping sides of a triangular trough 12 ft. long are inclined at 45° to the horizontal. Find the force on these sides and also on the ends when the water is 2 ft. deep.

12. Find the magnitude and the position of the resultant force of the water against the face of the dam shown in Fig. 138. Show that the

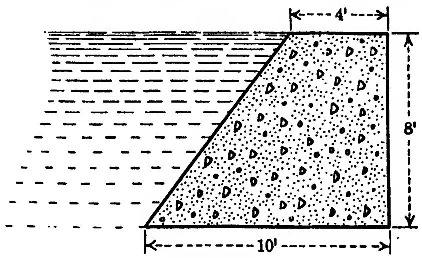

FIG. 138.

vertical component of this force is equal to the weight of the wedge of water resting on the dam. The length of the dam is 20 ft.

13. An opening in a dam has the form shown in Fig. 139. Where should a single horizontal prop be placed if it is to hold in place a plate which covers the opening? What compressive force would be exerted on the prop?

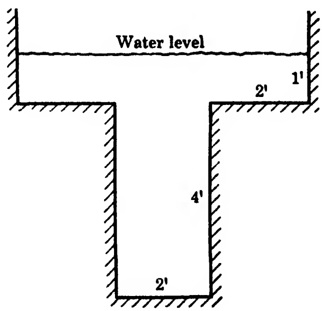

FIG. 139.

14. Show that the relation $y' = k^2/\bar{y}$ may also be written in the form $y' = I/M$ where I and M denote respectively the second and first moments of the area.

15. Show that the location of the center of pressure is given by the equation $y' = \bar{y} + \frac{k^2}{\bar{y}}$ where k denotes the radius of gyration of the area with respect to an axis through the centroid parallel to PP' (Fig. 136).

16. Suppose that a rectangular plate is submerged vertically with edge b in the surface of the water. Show that the center of pressure is at a depth $\frac{2}{3}h$ below the surface. Suppose now that the plate be pushed farther and farther beneath the surface. Show that the center of pressure remains below the centroid but approaches the centroid as the depth increases indefinitely.

135. Work.—Suppose that a constant force of F lb. acts upon a body and that the point of application of the force

moves a distance d ft. in the direction of the force. The product of the magnitude of the force and the distance moved is called the *work* done by the force on the body during the motion. Denoting it by U, we have

$$U = Fd \text{ ft.-lb.}$$

Thus, the work done by the 20 lb. force indicated in Fig. 140

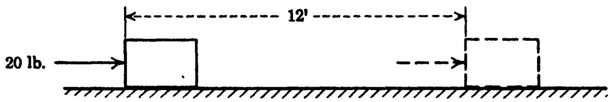

FIG. 140.

in pushing the block a distance of 12 ft. is $20(12) = 240$ ft.-lb.

136. Work done by a variable force.—We consider now the case in which the force does not remain constant throughout the distance interval but varies according to

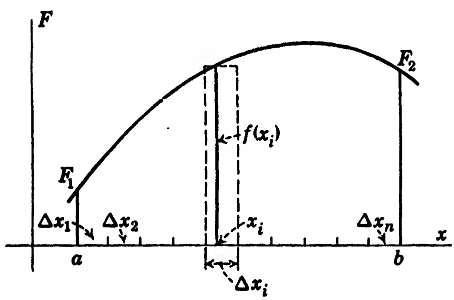

FIG. 141.

some definite law, say $F = f(x)$, from a value F_1 at the beginning to a value F_2 at the end of the interval. The way in which the force varies might be represented graphically as indicated in Fig. 141.

The work done by the variable force in the interval from $x = a$ to $x = b$ may be calculated as follows:

1. Divide the distance moved into a large number n of small intervals as indicated in Fig. 141.

2. The work done by the variable force in each small interval can be obtained, except for an infinitesimal of

higher order, by multiplying the length of the interval by the value of the force at *any* point in this interval; *i.e.*,

$$\Delta U_i = f(x_i)\Delta x_i.$$

3. The sum of these terms is a good approximation to U if each Δx is small, and the limit of the sum as each $\Delta x \to 0$ is exactly the required work. Hence,

$$U = \int_a^b f(x)dx.$$

It is immediately evident that the work done is represented geometrically by the area under the curve (Fig. 141) in the interval from $x = a$ to $x = b$.

Example 1

The force required to stretch a spring is directly proportional to the elongation, the force being 12 lb. when it is stretched 1 in. Find the work done in stretching it 8 in. beyond its free length.

Solution

As the spring is stretched, the applied force varies linearly from 0 to 96 lb. as indicated in Fig. 142. When it has been stretched x in. the force at that instant is $F = 12x$ lb. The work done is, then,

$$U = \int_0^8 12x\,dx = 384 \text{ in.-lb.}$$

It should be observed that this result may also be obtained by multiplying the total elongation of the spring by the value of the force at the mid-point of the stretch. Why?

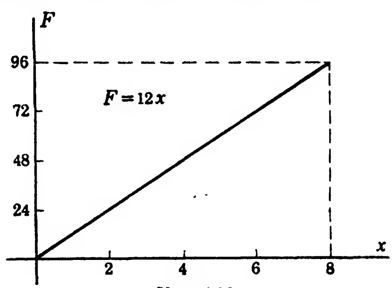

FIG. 142.

Example 2

The tank shown in Fig. 143 is full of oil weighing 50 lb. per cubic foot. Find the work done by a pump in emptying the tank if the oil is discharged at a point 10 ft. above the center of the tank.

Solution

We may divide the oil into a large number of slices as indicated in the figure. The force required to lift a slice is equal to its weight. The

work done on each slice is the product of this force and the distance through which the slice must be lifted. Hence, taking a slice at a distance y from the x-axis (assumed through the center of one end) we have,

Weight of slice $= 2\sqrt{4 - y^2}(12)(50)dy = 1{,}200\sqrt{4 - y^2}\,dy.$
Distance to be lifted $= 10 - y.$

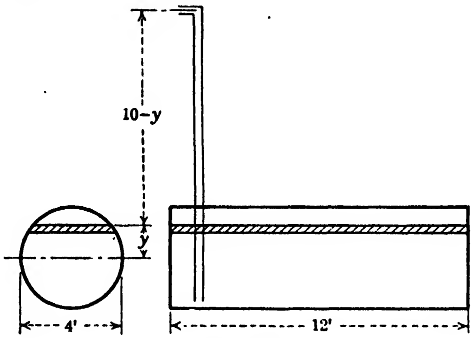

FIG. 143.

Work done on slice $= 1{,}200\sqrt{4 - y^2}(10 - y)dy.$
The total work done is obtained by integrating this expression between limits which cover all of the slices; *i.e.*, from $y = -2$ to $y = +2$. Thus,

$$\begin{aligned} U &= 1{,}200\int_{-2}^{+2}\sqrt{4 - y^2}(10 - y)dy \\ &= 24{,}000\pi \text{ ft.-lb.} \end{aligned}$$

The student should observe that it would not be permissible to integrate from 0 to 2 and double the result. Why?

PROBLEMS

1. Let $F(x)$ be the force applied to a spring when it has been stretched an amount x. Show that the work done in stretching it a small additional amount Δx differs from $F(x)\Delta x$ only by an infinitesimal of higher order.

2. Find the work done in stretching the spring of Example 1 an additional 8 in. if the force continues to obey the same law.

3. A certain spring requires a force of 10 lb. to stretch it $\frac{1}{2}$ in. Find the work done in stretching it 4 in. beyond its free length.

4. Find the work done in emptying half of the oil from the tank in Example 2.

5. A force acts on a body in the direction of motion for a distance of 9 ft. and varies in magnitude according to the law $F = 4\sqrt{x}$, x being the distance moved from the starting point. Sketch a curve showing the way in which F varies. Calculate the work done.

6. How would you define the *average value* of a variable force in an interval from $x = a$ to $x = b$? How would you calculate it? What is the average value of the force in Prob. 5?

7. Under what commonly occurring condition is the average value of a force which varies from F_1 at the beginning to F_2 at the end of an interval, equal to $\frac{1}{2}(F_1 + F_2)$?

8. A volume v_1 of steam is let into the cylinder of a steam engine under a pressure P_1. The steam expands, pushing the piston back, and at the end of the stroke the volume and pressure are v_2 and P_2. Show that the work done is $U = \int_{v_1}^{v_2} P\,dv$. HINT: The *force* against the piston is PA where A is its area. $U = \int F\,dx = \int PA\,dx$, and $A\,dx = dv$.

9. Calculate the work done in compressing 400 cu. ft. of air at 15 lb. per square inch to a volume of 30 cu. ft. assuming the law $Pv = C$.

10. A hemispherical vat 6 ft. in diameter is full of oil weighing 50 lb. per cubic foot. Calculate the work done in pumping it out if the oil is discharged through a pipe 4 ft. above the top of the vat.

11. Show that the result in Prob. 10 could be obtained by considering all of the oil concentrated at its centroid.

12. A horizontal cylindrical tank is 10 ft. long, 4 ft. in diameter, and is filled with water. Compute the amount of work required to empty it if the water is discharged at a point 6 ft. above the center of the tank. Solve in two ways.

13. A vat consists of a right circular cylinder with its axis vertical, to which is attached a hemispherical bottom. The cylinder is 8 ft. in diameter and 8 ft. high. Find the work done in emptying it if the liquid, which weighs w lb. per cubic foot, is discharged at the top of the tank.

14. The cross section of a cylindrical tank 12 ft. long is an ellipse with axes 6 and 4 ft., respectively. The tank lies with its axis and the major axis of the end horizontal. Set up an integral for the work done in emptying it if the liquid, which weighs w lb. per cubic foot, is discharged 4 ft. above the center of the tank.

15. A hemispherical vat 10 ft. in diameter is filled with a liquid weighing w lb. per cubic foot. Find the work done in lowering the level 3 ft. if the liquid is discharged at a point 5 ft. above the top of the vat.

CHAPTER XXIV

PARTIAL DERIVATIVES

137. Functions of two or more variables.—The abbreviation

$$z = f(x, y)$$

means that z is a function of the two variables x and y. Thus the volume of a right circular cylinder is a function

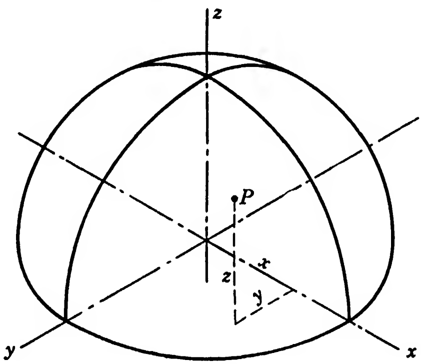

FIG. 144.

of its radius and height. In what follows we shall assume that in general x and y are independent of each other, although we shall consider certain cases in which the value of y depends upon that of x. The notation used here can be extended in an obvious way to the case of a function of any number of variables

From his study of solid analytic geometry, the student is familiar with the interpretation of the relation $z = f(x, y)$ as a surface. Thus, any point $P(x, y, z)$ in space whose coordinates satisfy the relation

$$z = \sqrt{16 - x^2 - y^2}$$

lies upon the surface of a hemisphere, as indicated in Fig. 144.

The ideas of single-valuedness, continuity, etc., which have been discussed for functions of one variable, are easily extended. The details are covered in the exercises of the next set. In the case of the above example, x and y are restricted to the "region" of the xy-plane for which $x^2 + y^2 \leqq 16$. For any such pair of values of x and y there is a *single* value of z. The function is continuous at every point inside this region and is therefore said to be continuous "over the region."

138. Partial derivatives.—Suppose that through any point P on the surface whose equation is $z = f(x, y)$, we

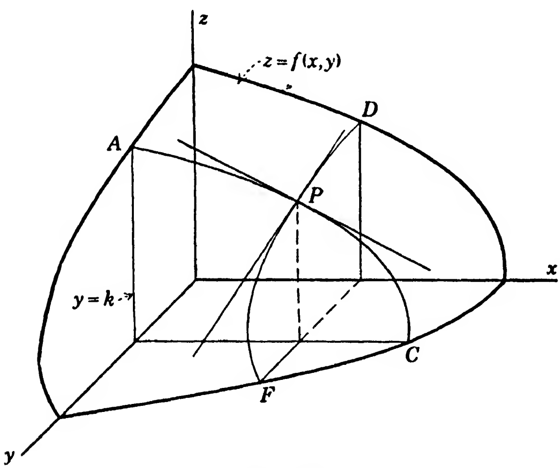

FIG. 145.

pass a plane parallel to the xz-plane. We thus cut from the surface a curve such that, as we move along it, x and z both change while *y remains constant.* Such a curve may be regarded as defined by the two equations

$$z = f(x, y), \qquad y = k. \qquad \text{(Fig. 145)}$$

The slope of the tangent line to this curve at P represents the rate at which z changes with respect to x *when x alone is allowed to vary,* y being held constant. It is called the *partial derivative* of the function *with respect to x* and is usually denoted by the symbol

$$\frac{\partial z}{\partial x}, \text{ or } \frac{\partial f(x, y)}{\partial x}, \text{ or simply } \frac{\partial f}{\partial x}.$$

In the same manner we may define the corresponding partial derivative of the function with respect to y. Its value at P represents the slope of the corresponding curve cut from the surface by a plane parallel to the yz-plane.

Either of these partial derivatives is found by differentiating the relation $z = f(x, y)$ in the usual manner with respect to this variable, *treating the other variable as if it were a constant.*

Example 1

$$z = x^3 - 2x^2y + xy^2 + y^3.$$
$$\frac{\partial z}{\partial x} = 3x^2 - 4xy + y^2;$$
$$\frac{\partial z}{\partial y} = -2x^2 + 2xy + 3y^2.$$

Example 2

$$f(x, y) = \log (x^2 + y^2).$$
$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2};$$
$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}.$$

Example 3

The radius and the height of a right circular cone are 8 and 12 in., respectively. Find the rate of increase of the volume with respect to the radius at this instant if the radius alone is increasing, the height remaining constant.

Solution

$$v = \tfrac{1}{3}\pi r^2 h.$$
$$\frac{\partial v}{\partial r} = \tfrac{2}{3}\pi r h$$
$$\left.\frac{\partial v}{\partial r}\right]_{\substack{r=8\\h=12}} = 64\pi \text{ cu. in. per inch.}$$

PROBLEMS

1. What is meant by saying that a certain function $f(x, y)$ is single-valued over a certain region of the xy-plane?

2. Make a sketch to illustrate the statement: $f(x, y)$ is continuous at (x_1, y_1) if

$$|f(x_1, y_1) - f(x, y)| < \epsilon \text{ provided } \begin{cases} |x_1 - x| < \delta_1 \\ |y_1 - y| < \delta_2. \end{cases}$$

3. Make a sketch to illustrate the following formal definition of the partial derivative:

$$\frac{\partial f(x, y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Write down a corresponding definition of $\partial f/\partial y$.

4. Sketch the surface $4z = x^2 + y^2$. Evaluate $\partial z/\partial y$ at (2, 4, 5) and show its geometrical significance.

5. At $P(2, \sqrt{5}, 4)$ on the sphere $x^2 + y^2 + z^2 = 25$, a tangent line is drawn parallel to the xz-plane. Find its slope. Why should the result be negative?

6. Sketch one octant of the ellipsoid $\frac{x^2}{6} + \frac{y^2}{3} + \frac{z^2}{2} = 1$. Show the curves cut from it by the planes $x = 1$ and $y = 1$. Find the slopes of these curves at $P(1, 1, 1)$.

7. Find the slope of the line parallel to the xz-plane which is tangent to the surface $xyz = 8$ at (2, 2, 2).

In each of the following, find $\partial z/\partial x$ and $\partial z/\partial y$:

8. $z = ye^x + xe^y$.

9. $z = x^3 + 3xy^2 - 2y^3$.

10. $z = \log (x^2 - y^2)$.

11. $z = \dfrac{y^2}{y - x}$.

12. $z = x^2y - \sin xy$.

13. $z = \text{arc} \sin \dfrac{y}{x}$.

14. $x^2 + y^2 + z^2 = 16$.

15. If $U = \frac{1}{2}xy \sin \theta$ find the three first partial derivatives of U.

16. If $z = \dfrac{x^3 - y^3}{xy}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

17. Write the equations of the tangent line to the parabola $z = 2x^2 + y^2$, $y = 2$, at $P(1, 2, 6)$.

18. A line is drawn through the point (1, 2, 2) parallel to the xz-plane and tangent to the sphere $x^2 + y^2 + z^2 = 9$. Write its equations.

19. Two sides and the included angle of a triangle are respectively x, y, and θ. Find the rate of change of its area with respect to each of these when the other two are held constant.

20. The pressure, volume, and temperature of a confined gas are connected by the relation $pv = ct$ where c is a constant. Find the rate of change of the pressure with respect to the volume when the temperature

is held constant and the rate of change of the pressure with respect to the temperature when the volume remains constant.

139. Tangent plane and normal line.—Let $P(x_1, y_1, z_1)$ be any point on the surface whose equation is $z = f(x, y)$. The equation of *any* plane through P may be written in the form

$$z - z_1 = A(x - x_1) + B(y - y_1).$$

That this equation represents a plane through P, no matter what values are assigned to A and B, is clear from the fact

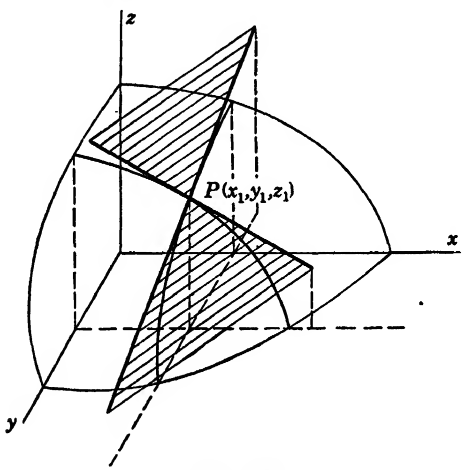

FIG. 146.

that it is a linear equation in x, y, and z, and is satisfied by the coordinates of P.

We wish now to determine A and B so that the above equation will represent the *tangent plane* to the surface at P. For this purpose we note that if the plane is tangent to the surface, the slopes of the lines cut from the plane by the planes $y = y_1$ and $x = x_1$ are equal respectively to the values of $\partial z/\partial x$ and $\partial z/\partial y$ at P. Since the slopes of these lines are A and B respectively we have, for tangency,

$$A = \frac{\partial z}{\partial x}\bigg]_P \qquad B = \frac{\partial z}{\partial y}\bigg]_P.$$

The equation of the tangent plane at P is, then,

$$\text{(I)} \qquad z - z_1 = \left.\frac{\partial z}{\partial x}\right]_P (x - x_1) + \left.\frac{\partial z}{\partial y}\right]_P (y - y_1).$$

The line through P perpendicular to the tangent plane is called the *normal line* to the surface. The student will recall from his study of analytic geometry that the coefficients A, B, and C, in an equation of the form

$$Ax + By + Cz + D = 0$$

are direction numbers of the *normal* to the plane represented by the equation. It is obvious that if (I) were written in this form, the coefficients of x, y, and z would be $\left.\frac{\partial z}{\partial x}\right]_P$, $\left.\frac{\partial z}{\partial y}\right]_P$, and -1 respectively. Hence the direction numbers of the normal to the tangent plane at P are

$$\left.\frac{\partial z}{\partial x}\right]_P, \qquad \left.\frac{\partial z}{\partial y}\right]_P, \text{ and } -1.$$

Since the equations of a line through $P(x_1, y_1, z_1)$ with direction numbers l, m, and n, are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n},$$

we have for the equation of the normal line to the surface at P

$$\text{(II)} \qquad \frac{x - x_1}{\left.\frac{\partial z}{\partial x}\right]_P} = \frac{y - y_1}{\left.\frac{\partial z}{\partial y}\right]_P} = \frac{z - z_1}{-1}.$$

Example

Find the equations of tangent plane and normal line to the sphere $x^2 + y^2 + z^2 = 17$ at $P(3, -2, 2)$.

Solution (Fig. 147)

$$\frac{\partial z}{\partial x} = -\frac{x}{z}; \qquad \left.\frac{\partial z}{\partial x}\right]_{3,-2,2} = -\frac{3}{2}.$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}; \qquad \left.\frac{\partial z}{\partial y}\right]_{3,-2,2} = 1.$$

The equation of the tangent plane is, then,

$$z - 2 = -\tfrac{3}{2}(x - 3) + 1(y + 2),$$

or

$$3x - 2y + 2z = 17.$$

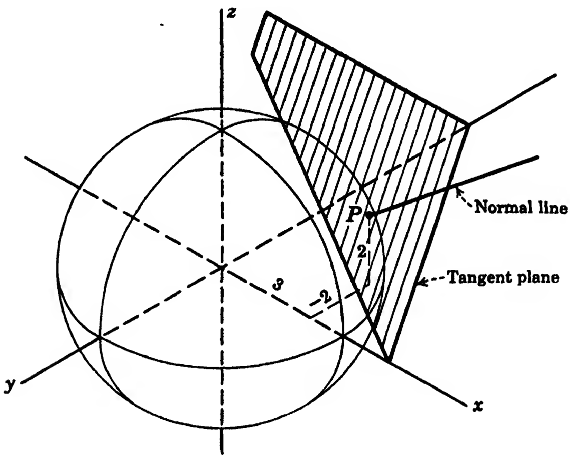

FIG. 147.

The equations of the normal line are

$$\frac{x - 3}{-\frac{3}{2}} = \frac{y + 2}{1} = \frac{z - 2}{-1},$$

or

$$\begin{cases} 2x + 3y = 0 \\ y + \quad z = 0. \end{cases}$$

PROBLEMS

Find the equations of tangent plane and normal line to each of the following surfaces at the point indicated:

1. $x^2 + y^2 = 4z$ at $(2, 2, 2)$.
2. $z + 2 = x^2 + y^2$ at $(2, 1, 3)$.
3. $x^2 + y^2 + z^2 = 38$ at $(5, 2, 3)$.
4. $xyz = 16$ at $(4, 2, 2)$.
5. $ax + by + cz = d$ at (x_1, y_1, z_1).
6. $\dfrac{x^2}{16} + \dfrac{y^2}{8} + \dfrac{z^2}{4} = 1$ at $(-2, 2, 1)$.
7. Show that the tetrahedron bounded by the coordinate planes and any tangent plane to the surface $xyz = a^3$ is of constant volume.
8. Show that the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = r^2$ at $P(x_1, y_1, z_1)$ is $x\,x_1 + y\,y_1 + z\,z_1 = r^2$.

9. Show that the equation of the tangent plane to the ellipsoid $ax^2 + by^2 + cz^2 = d$ at $P(x_1, y_1, z_1)$ is $ax_1x + by_1y + cz_1z = d$.

10. Find the equation of the tangent plane to the hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ at $P(x_1, y_1, z_1)$.

140. Total differential.—In Chap. XIII we defined the quantity $f'(x)\Delta x$ as the *differential* of the function $f(x)$.

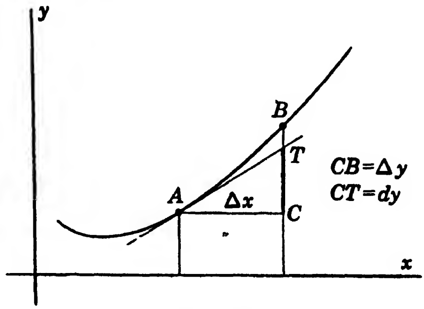

FIG. 148.

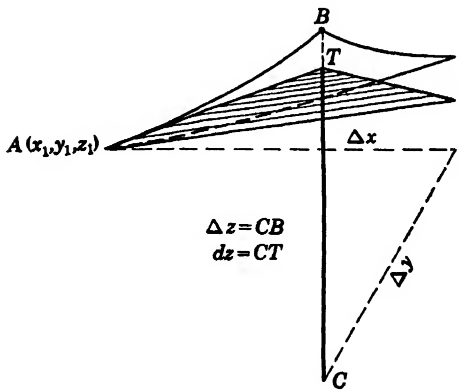

FIG. 149.

It was useful in finding *approximately* the change in the value of $f(x)$ caused by a small change Δx in x. Since, in Fig. 148, $\Delta y = CB$ while $f'(x)\Delta x = CT$, this amounts to using *the increment in the ordinate to the tangent line* as an approximation to Δy.

In the case of a function of two variables the situation is entirely analogous. Suppose that, starting at $A(x_1, y_1, z_1)$ on the surface $z = f(x, y)$ (Fig. 149), we let x and y both

change by small amounts Δx and Δy. The change produced in the value of the function is represented by

$$\Delta z = CB = f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1, y_1).$$

As an approximation to Δz we may use the part represented by CT which is *the increment in the ordinate to the tangent plane.* The equation of this plane is

$$z - z_1 = \left.\frac{\partial z}{\partial x}\right]_A (x - x_1) + \left.\frac{\partial z}{\partial y}\right]_A (y - y_1),$$

and if we let $x - x_1 = \Delta x$ and $y - y_1 = \Delta y$, we have immediately

$$CT = \left.\frac{\partial z}{\partial x}\right]_A \Delta x + \left.\frac{\partial z}{\partial y}\right]_A \Delta y.$$

The student will see easily from the figure that CT is a good approximation to Δz if Δx and Δy are small. It can be shown that $CT/\Delta z \to 1$ as Δx and $\Delta y \to 0$; *i.e.*, the difference (represented by TB) is an infinitesimal of higher order than Δz and represents therefore an *arbitrarily small fraction of* Δz, when Δx and Δy are sufficiently small.

This approximation to Δz is called the *total differential* of the function z; it is also called the *principal part* of the increment Δz. Denoting it by dz we have

$$\text{(III)} \qquad \boldsymbol{dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.}$$

Taking the above equation as the definition of the total differential of *any* function and applying it to the particular functions

$$f(x, y) = x \qquad \text{and} \qquad \varphi(x, y) = y,$$

we find that

$$dx = \Delta x \qquad \text{and} \qquad dy = \Delta y.$$

That is, the differentials of the *independent* variables are equal to their increments. We may then write (III) in

the form

(IV) $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

For a function of three or more variables it can be shown that a similar formula holds. Thus, if $U = f(x, y, z)$, the quantity

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

gives the principal part of the total change in U caused by changes dx, dy, dz, in the independent variables x, y, and z.

We use the total differential of a function of several variables as an approximation to its increment, just as we used the differential of a function of a single variable.

Example

The metal in a cylindrical can is $\frac{1}{64}$ in. thick on top, bottom, and sides. The inside dimensions are $r = 4$ in., $h = 8$ in. Find approximately the volume of the metal.

Solution

The volume of metal is equal to the amount by which the volume of a cylinder increases when its radius changes from 4 to $4\frac{1}{64}$ in. and its height from 8 to $8\frac{1}{32}$ in. Why?

$$\begin{aligned} V &= \pi r^2 h, \\ dV &= \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh \\ &= 2\pi hr\,dr + \pi r^2 dh. \end{aligned}$$

Taking $r = 4$, $h = 8$, $dr = \frac{1}{64}$, $dh = \frac{1}{32}$, we have

$$\begin{aligned} dV &= 64\pi(\tfrac{1}{64}) + 16\pi(\tfrac{1}{32}) \\ &= \tfrac{3}{2}\pi \text{ cu. in.} \end{aligned}$$

PROBLEMS

1. (*a*) Show that if $z = x^2y$, $dz = 2xy\,dx + x^2dy$.

(*b*) Use the result of (*a*) to find approximately $(5.04)^2(2.98)$.

(*c*) Multiply $(5 + 0.04)^2(3 - 0.02)$. Which of these terms are neglected in the approximation obtained in (*b*)?

2. Find the differential of x^2y^3 and use it to find approximately $(5.06)^2(2.02)^3$.

3. Find the total differential of the function $U = xyz$. Multiply out $(x + dx)(y + dy)(z + dz)$. What terms constitute the difference between dU and ΔU?

4. By approximately how much does $\sqrt{xy}$ change when x changes from 8 to 8.14 and y from 2 to 1.98? Compute approximately $\sqrt{(8.14)(1.98)}$.

Write down the total differentials of the following functions:

5. $z = x^2 + y^2$.

6. $z = 3x^2 + 6xy + 4y$.

7. $z = \dfrac{1}{xy}$.

8. $z = \dfrac{x}{1 - y^2}$.

9. $z = \dfrac{\tan x}{y^2}$.

10. $U = \frac{1}{2}xy \sin \theta$.

11. $Q = x^2 + y^2 - 2xy \cos \theta$.

12. A metal box is 8 ft. long, 4 ft. wide, and 2 ft. deep. The thickness of sides and top is 0.05 ft. and that of the bottom is 0.15 ft. Find approximately the volume of metal in the box. Compare this with the exact volume. See Prob. 3.

13. The diameter and height of a cylindrical can are each 8 in. with a possible error of $\frac{1}{64}$ in. in each. What is approximately the maximum possible error in the computed volume? In the lateral surface? Compute the percentage errors.

14. The specific gravity of a solid is found from the formula $S = \dfrac{A}{A - W}$ where A and W are its weight in air and water respectively. If the readings are $A = 6.4$ lb., $W = 2.8$ lb. with possible errors of 0.05 lb. in each, what is the possible error in S?

15. A circular sector is constructed with central angle 30° and radius 10 in. The angle may be in error by as much as $\frac{1}{10}°$ and the radius by $\frac{1}{64}$ in. By how much may the actual area differ from the computed value?

16. A triangle is staked out, two sides and the included angle being respectively 80 ft., 120 ft., and 60°. There are possible errors of 3 in. in each measured side and $\frac{1}{2}°$ in the angle. By how much may the actual area differ from the computed value?

17. The time recorded for a sprinter in the 100-yd. dash is 10 sec. with a possible error of 0.1 sec. What is the uncertainty in the computed average speed if the distance may be in error by as much as 4 in.?

18. Show that the relative error in the volume of a rectangular box is equal to the sum of the relative errors in its three edges.

19. Errors of 2 per cent and 1 per cent are made in measuring the radius and height respectively of a right circular cone. Find the percentage error in the computed volume.

20. Show that the relative error in the volume of a right circular cone is equal to twice that in the radius plus that in the height.

21. Show that the relative error in a function of the form Kx^my^n is equal to m times the relative error in x plus n times the relative error in y.

22. If two resistors having resistances r_1 and r_2 are used in parallel, the resulting resistance R is given by the relation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Find the percentage error in the computed value of R due to errors of 1 per cent in r_1 and r_2.

141. Total derivative—rates.—Consider the relation

$$z = f(x, y)$$

and suppose that x and y vary continuously with the time in accordance with the relations

$$x = \alpha(t) \qquad y = \beta(t).$$

Under these conditions z becomes a function of the single independent variable t and we wish to find dz/dt.

The student will recall that if $z = \Psi(t)$, dz/dt can be found by dividing the *principal part* of Δz by $\Delta t = dt$. This principal part of Δz was found in Art. 140 to be given by

$$\text{(IV)} \qquad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

If x and y are independent variables; in this case dx and dy represent the actual increments given to x and y. It can be shown that in the case now under discussion, the principal part of Δz is still given by (IV), dx and dy now representing the *principal parts* of the increments in x and y caused by an arbitrary increment $\Delta t = dt$ in t.

Assuming this, we may divide each side of (IV) by dt and we have

$$\text{(V)} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

It can be shown that a similar formula holds in the case of a function of n variables. Suppose, for example, that

$U = f(x, y, z)$ and that, at a certain instant, x, y, and z are changing at rates dx/dt, dy/dt, and dz/dt. The rate of change of U is then

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}.$$

Example

The edges of a rectangular box are $x = 20$ in., $y = 12$ in., $z = 8$ in.; x and y are increasing at 2 in. per minute and z is decreasing at 3 in. per minute. At what rate is the volume changing?

Solution

$$v = xyz$$

$$\frac{\partial v}{\partial x} = yz, \qquad \frac{\partial v}{\partial y} = xz, \qquad \frac{\partial v}{\partial z} = xy;$$

$$\frac{dv}{dt} = yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt}.$$

At this instant,

$$x = 20, \qquad y = 12, \qquad z = 8, \qquad \frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = 2, \qquad \frac{dz}{dt} = -3.$$

We have, then,

$$\frac{dv}{dt} = 96(2) + 160(2) + 240(-3) = -208 \text{ cu. in. per minute.}$$

The negative sign indicates that the volume is decreasing.

In this discussion we have been thinking of t as representing time. It may however be any independent variable upon which x and y depend. Geometrically, the parametric equations $x = \alpha(t)$, $y = \beta(t)$, define a curve CD (Fig. 150) in the xy-plane. Consequently, the equations

$$z = f(x, y), \qquad x = \alpha(t), \qquad y = \beta(t)$$

define the curve AB cut from the surface $z = f(x, y)$ by the cylindrical surface whose elements intersect CD and are perpendicular to the *xy-plane.* The value of dz/dt is the rate at which z changes as we move along this curve—***measured with respect to t.***

If x is the independent variable, the curve AB in Fig. 150 is represented by the equations

$$z = f(x, y), \qquad y = \varphi(x).$$

Dividing both sides of (IV) by dx we have

$$\text{(VI)} \qquad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

This is called the *total derivative* of z with respect to x. The student should note carefully the difference between this derivative and the *partial* derivative of z with respect

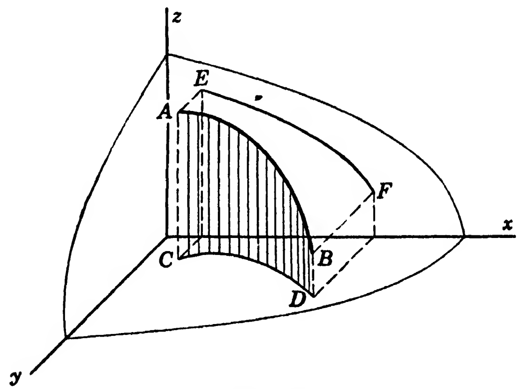

FIG. 150.

to x. The partial derivative measures the rate of change of z with respect to x when x alone changes, *y being held constant.* In the total derivative, x and y both vary, the change in y bearing a definite relation to that in x. The student should show that dz/dx represents the slope of the curve EF which is the projection of AB on the xz-plane.

142. Directional derivative.—At a point P on the surface $z = f(x, y)$, the values of $\partial z/\partial x$ and $\partial z/\partial y$ are the slopes of the tangent lines drawn to the surface at P, parallel to the xz- and yz-planes respectively. We consider now the problem of finding the slope of *any* tangent line to the surface at P.

Let the direction of any tangent line be specified by the angle α which its projection on the xy-plane makes with

the x-axis as indicated in Fig. 151. The slope of this tangent line is found by dividing dz, not by dx, but by $ds = dx \sec \alpha$. Denoting the result by dz/ds, we have

$$\tan \theta = \frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{dx \sec \alpha} + \frac{\partial z}{\partial y}\frac{dy}{dx \sec \alpha}.$$

Since $dx/dx = 1$ and $dy/dx = \tan \alpha$, this reduces to

$$\text{(VII)} \qquad \frac{dz}{ds} = \frac{\partial z}{\partial x}\cos \alpha + \frac{\partial z}{\partial y}\sin \alpha.$$

This expression gives the slope of any tangent line to the surface. Its value at P represents the rate at which z

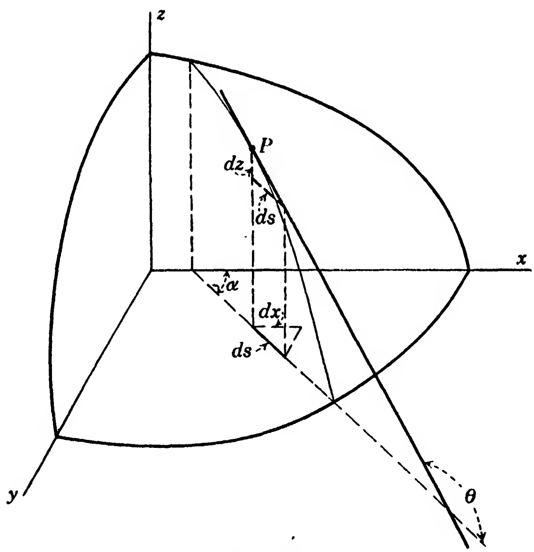

FIG. 151.

changes when we move in any direction from P, the rate being measured with respect to the horizontal distance moved in the corresponding direction by the projection of P on the xy-plane. For $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$ respectively, it of course reduces to $\partial z/\partial x$ and $\partial z/\partial y$.

We may now find the slope of the steepest tangent line to the surface at P. For this purpose we note that when we consider any definite point P on the surface, the values of

$\partial z/\partial x$ and $\partial z/\partial y$ are *fixed* and the right-hand side of (VII) is a function of α alone. In order to find the value of α for which this expression has a maximum value we may set its derivative with respect to α equal to zero. Thus,

$$-\frac{\partial z}{\partial x}\sin\alpha + \frac{\partial z}{\partial y}\cos\alpha = 0,$$

$$\tan\alpha = \frac{\dfrac{\partial z}{\partial y}}{\dfrac{\partial z}{\partial x}}.$$

Substituting this value of α in (VII) we find that the extreme value of dz/ds is

$$\left|\frac{dz}{ds}\right|_{max.} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

PROBLEMS

1. The edges of a piece of copper which has the form of a rectangular parallelepiped are 2, 4, and 8 in. Because of heating, the edges are increasing at 0.001 in. *per inch* per minute. At what rate is the volume changing?

2. Two sides and the included angle of a triangle are 10 in., 20 in., and 30°. If the sides are increasing at 1 in. per minute, at what rate must the angle decrease if the area remains constant?

3. Sand is being poured upon a conical pile at 4 cu. ft. per minute. The pile is 6 ft. in diameter and 6 ft. high. At what rate is its height increasing if its diameter is increasing at 2 in. per minute?

4. A point moves on the sphere $x^2 + y^2 + z^2 = 36$. When it goes through $P(4, 2, 4)$, its x- and y- coordinates are increasing at 6 units per minute. At what rate is z changing? What is the direction of its path at this instant? Draw a figure.

5. Sketch carefully the first octant of the sphere $x^2 + y^2 + z^2 = 36$. Show the curve AB cut from the surface by the plane $y = 2x$. Show the point $P(2, 4, 4)$ on this curve. Then,

(*a*) Evaluate dz/dx at P by two methods. Show the geometrical significance of the result.

(*b*) Find the slope of the curve AB at P.

6. A line is drawn so as to intersect the z-axis and be tangent to the ellipsoid $x^2 + 2y^2 + z^2 = 25$, at $P(4, 2, 1)$. Find its slope.

7. Show that at any point $P(x, y, z)$ on the sphere $x^2 + y^2 + z^2 = r^2$, the steepest tangent line is the one which intersects the z-axis.

8. Show that at $P(x, y, z)$ on the plane $Ax + By + Cz + D = 0$, the steepest line which can be drawn in the plane is perpendicular to the xy-trace of the plane.

9. Find the slope of the curve cut from the paraboloid $6z = x^2 + y^2$ by the plane $2y = x + 3$ at $P(3, 3, 3)$.

10. A small marble is placed at $P(4, 1, 4)$ on the plane

$$2x + 4y + 3z = 24$$

and allowed to roll down. If the xy-plane is horizontal, at what point will the marble strike it?

11. Find the slope of the curve cut from the ellipsoid

$$x^2 + 4y^2 + 2z^2 = 16$$

by the plane $x + y = 3$, at $P(2, 1, 2)$. Draw the figure.

12. Sketch the paraboloid $x^2 + y^2 + z = 16$ and show the point $P(3, 2, 3)$ on it. Show that the steepest tangent line to the surface at P is the one which intersects the z-axis and find its slope.

13. Show the curve cut from the surface of Prob. 12 by the plane $3x + 4y = 12$. Find its slope at the point where $x = 1$.

14. As the radius of a right circular cylinder increases the height also increases, the relation between them always being $h = r^2 + 2$. Find the total derivative of v with respect to r both with and without using formula (VI). Explain the difference between this and $\partial v/\partial r$.

15. The radius of a right circular cone increases at $\frac{1}{2}$ in. per minute, and the height is always equal to $r^2 + 4$. Find dv/dr and dv/dt when $r = 6$ in. State the units in which each is expressed.

143. Differentiation of implicit functions.—From a relation $f(x, y) = 0$ we have already found dy/dx by the method of implicit differentiation. An equivalent method is derived as follows: Consider the relation $f(x, y) = 0$ and for the moment let $z = f(x, y)$ where $z \equiv 0$. Then, by (VI),

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

Now, since $z \equiv 0$, $dz/dx \equiv 0$. Making this substitution and solving for dy/dx we have

$$\text{(VIII)} \qquad \frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \qquad \left(\text{if } \frac{\partial f}{\partial y} \neq 0\right).$$

Example

Find dy/dx from the relation $x^2 + y^2 - 3xy = 0$.

Solution

Let

$$f(x, y) = x^2 + y^2 - 3xy,$$

then,

$$\frac{\partial f}{\partial x} = 2x - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y - 3x.$$

Using (VIII), we have

$$\frac{dy}{dx} = \frac{3y - 2x}{2y - 3x}.$$

Now let z be defined implicitly as a function of x and y by the relation $f(x, y, z) = 0$, and consider the problem of finding $\partial z/\partial x$ *without first solving for* z. Since in finding this derivative y is held constant, $f(x, y, z)$ is a function of x and z only, and from (VIII) we have

$$\text{(IX)} \qquad \frac{\partial z}{\partial x} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}} \qquad \left(\text{if } \frac{\partial f}{\partial z} \neq 0\right).$$

Example

Find $\partial z/\partial x$ from the relation $x^2 + y^2 + z^2 = 16$.

Solution

Let

$$f(x, y, z) = x^2 + y^2 + z^2 - 16.$$

$$\frac{\partial f}{\partial x} = 2x; \qquad \frac{\partial f}{\partial z} = 2z.$$

Using (IX), we find

$$\frac{\partial z}{\partial x} = -\frac{x}{z}.$$

An exactly analogous formula, of course, holds for $\partial z/\partial y$.

144. Higher partial derivatives.—The derivatives $\partial z/\partial x$ and $\partial z/\partial y$ of a function $z = f(x, y)$ are themselves, in

general, functions of x and y and can be differentiated again with respect to either x or y. The following notation is used for these second partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y),$$
$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y),$$
$$\left\{\begin{aligned} \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial y\, \partial x} = f_{xy}(x, y), \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial^2 z}{\partial x\, \partial y} = f_{yx}(x, y). \end{aligned}\right.$$

It thus appears that there are four different second partial derivatives of a function of two variables. It can be shown, however, that the last two are identical for all values of x and y for which they are continuous; *i.e.*, if one differentiates $f(x, y)$ with respect to x and then differentiates the result with respect to y, he obtains the same result as if he had performed the differentiations in the reverse order.

The notation used above can easily be extended to derivatives of higher order. Thus, there are four partial derivatives of $f(x, y)$ of third order; namely,

$$\frac{\partial^3 f}{\partial x^3}, \qquad \frac{\partial^3 f}{\partial x^2 \partial y}, \qquad \frac{\partial^3 f}{\partial x\, \partial y^2}, \qquad \frac{\partial^3 f}{\partial y^3}.$$

The symbol $\partial^3 f/\partial x\, \partial y^2$ denotes the result of differentiating $f(x, y)$ successively twice with respect to y and once with respect to x. The order in which these differentiations are performed is immaterial.

PROBLEMS

Find dy/dx by two methods, without first solving for y:

1. $x^2 + y^2 = 4$. **2.** $xy = 16$.
3. $xy + 2y - 3x = 1$. **4.** $x^2 + y^2 - 3axy = 0$.
5. $y \sin x = x \cos y$. **6.** $x^2 + 3xy = y^2 - 8$.

Find $\partial z/\partial x$ and $\partial z/\partial y$:

7. $x^2 + y^2 - 4z^2 = 16.$

8. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

9. $xyz = 10.$

10. $x^3 + y^3 + z^3 - 3xyz = 0.$

Find the second partial derivatives of the following functions:

11. $f(x, y) = 3x^2 - xy + 2y^2.$

12. $\varphi(x, y) = x^3 + 4xy^2 + x^2y + y^3.$

13. $f(x, y) = e^{x+y}.$

14. $f(x, y) = \dfrac{xy}{x + y}.$

15. Verify that $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$ for the function

$$z = 4x^3 + 3x^2y - 3y^2.$$

16. $z = x^3y^2 - 4xy^4 + 4x^2y^3$. Find $\partial^3 z/\partial x^2\, \partial y$, using three different orders of differentiation.

17. $z = \log \sqrt{x^2 + y^2}$. Show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

18. $U = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$. Show that $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2} = 0.$

CHAPTER XXV

MULTIPLE INTEGRALS

145. Definite double integral.—The symbol

$$\int_{x=a}^{x=b}\left[\int_{y=c}^{y=d} f(x, y)dy\right]dx, \qquad \text{or simply} \qquad \int_a^b\int_c^d f(x, y)dy\,dx$$

is called a *definite double integral.* Its value is defined in terms of two successive integrations as follows: First integrate $f(x, y)$ with respect to y between the limits $y = c$ and $y = d$; in this integration x is treated as a constant. Then, integrate the result with respect to x between the limits $x = a$ and $x = b$.

Example

Evaluate $\int_1^2\int_0^3 (x^2 + y^2)dy\,dx$.

Solution

Integrating with respect to y between limits $y = 0$ and $y = 3$ we have

$$\int_1^2\int_0^3 (x^2 + y^2)dy\,dx = \int_1^2 x^2y + \frac{y^3}{3}\Big]_{y=0}^{y=3} dx$$
$$= \int_1^2 (3x^2 + 9)dx.$$

Integrating next with respect to x gives

$$= x^3 + 9x\Big]_1^2$$
$$= 16.$$

In applied problems the limits c and d are often functions of x instead of constants.

Example

Evaluate $\int_0^1\int_x^{x^2} (x^2 + 3y + 2)dy\,dx$.

Solution

$$\int_0^1\int_x^{x^2}(x^2+3y+2)dy\,dx = \int_0^1 x^2y+\frac{3y^2}{2}+2y\Big]_x^{x^2}dx$$
$$= \int_0^1\left(\frac{5x^4}{2}-x^3+\frac{x^2}{2}-2x\right)dx$$
$$= \frac{x^5}{2}-\frac{x^4}{4}+\frac{x^3}{6}-x^2\Big]_0^1$$
$$= -\frac{7}{12}.$$

146. Definite multiple integral.—The notation just used can easily be extended to functions of any number of variables. Thus, the symbol

$$\int_a^b\int_c^d\int_e^f f(x,\ y,\ z)dz\,dy\,dx$$

means that $f(x,\ y,\ z)$ is to be integrated with respect to z between the limits $z = e$ and $z = f$, x and y being treated as constants; the result is to be integrated with respect to y between the limits $y = c$ and $y = d$, x being treated as a constant; and finally this result is to be integrated with respect to x between the limits $x = a$ and $x = b$.

Example

Evaluate $\int_2^4\int_1^2\int_0^1(xyz)dz\,dy\,dx.$

Solution

$$\int_2^4\int_1^2\int_0^1(xyz)dz\,dy\,dx = \int_2^4\int_1^2\frac{xyz^2}{2}\Big]_0^1 dy\,dx$$
$$= \frac{1}{2}\int_2^4\int_1^2 xy\,dy\,dx$$
$$= \frac{1}{2}\int_2^4\frac{xy^2}{2}\Big]_1^2 dx$$
$$= \frac{3}{4}\int_2^4 x\,dx$$
$$= \frac{3}{4}(6) = \frac{9}{2}.$$

In later applications the limits on $z(e$ and $f)$ often are functions of x and y, and the limits on $y(c$ and $d)$ may be functions of x.

Example

Evaluate $\int_0^1 \int_0^x \int_0^{x+y} dz\, dy\, dx.$

Solution

In this case $f(x, y, z) \equiv 1$, and since $\int(1)dz = z$ we have

$$\int_0^1 \int_0^x \int_0^{x+y} dz\, dy\, dx = \int_0^1 \int_0^x z\Big]_0^{x+y} dy\, dx$$
$$= \int_0^1 \int_0^x (x + y) dy\, dx$$
$$= \int_0^1 xy + \frac{y^2}{2}\Big]_0^x dx$$
$$= \int_0^1 \tfrac{3}{2}x^2 dx = \tfrac{1}{2}.$$

The student should note carefully that, in this book, limits on the *inside* integral sign refer to the variable indicated by the *inside* differential. In some textbooks the opposite convention is used.

PROBLEMS

Evaluate the following integrals:

1. $\int_2^4 \int_1^2 x\, dy\, dx.$
2. $\int_0^3 \int_{-2}^{-5} (x^2 + y^2) dy\, dx.$
3. $\int_0^\pi \int_0^{2 \sin \theta} \rho\, d\rho\, d\theta.$
4. $\int_0^r \int_0^{\sqrt{r^2 - x^2}} dy\, dx.$
5. $\int_0^{\frac{\pi}{2}} \int_0^{\sin \theta} \rho \cos \theta\, d\rho\, d\theta.$
6. $\int_0^1 \int_{-t}^t (z + t) dz\, dt.$
7. $\int_1^2 \int_2^4 \int_0^1 x\, dx\, dy\, dz.$
8. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^4 d\rho\, d\theta\, d\varphi.$
9. $\int_0^1 \int_0^x \int_0^{x+y} xyz\, dz\, dy\, dx.$
10. $\int_0^1 \int_0^{1-x} \int_0^{1-y^2} z\, dz\, dy\, dx.$
11. $\int_0^{2\pi} \int_0^\pi \int_0^r \rho^2 \sin \varphi\, d\rho\, d\varphi\, d\theta.$
12. $8\int_0^r \int_0^{\sqrt{r^2 - x^2}} \int_0^{\sqrt{r^2 - x^2 - y^2}} dz\, dy\, dx.$

147. Area by double integration.—We may easily express as a double integral the area $PQRS$ bounded by the non-intersecting curves $y = f(x)$ and $y = \varphi(x)$, and the ordinates at $x = a$ and $x = b$. For, from previous considerations (Fig. 152),

$$
\begin{aligned}
PQRS &= aPQb - aSRb \\
&= \int_a^b f(x)dx - \int_a^b \varphi(x)dx \\
&= \int_a^b [f(x) - \varphi(x)]dx.
\end{aligned}
$$

But this last integral is exactly the same as $\int_a^b \int_{\varphi(x)}^{f(x)} dy\, dx$; hence

$$PQRS = \int_a^b \int_{\varphi(x)}^{f(x)} dy\, dx.$$

In order to interpret this double integration as a double summation process we may divide the area into small

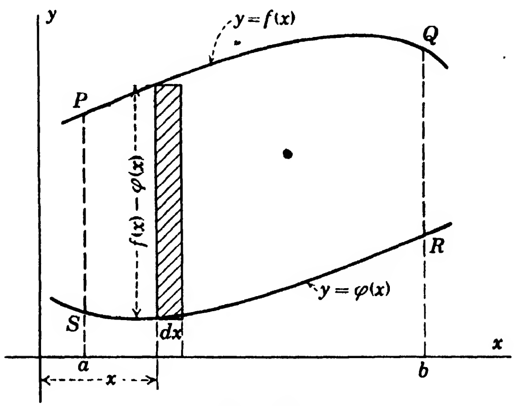

FIG. 152.

rectangular *elements* with bases Δx and altitudes Δy (plus some little irregular areas) by drawing lines parallel to the axes as shown in Fig. 153. The first of the two integrations (with respect to y) amounts to summing up all of the elements which lie in a vertical strip and taking the limit of the sum as $\Delta y \to 0$, x and Δx remaining fixed; *i.e.*,

$$
\begin{aligned}
\left[\int_{\varphi(x)}^{f(x)} dy\right] \cdot \Delta x &= [f(x) - \varphi(x)] \cdot \Delta x \\
&= \text{area } klmn \\
&= \lim_{\Delta y \to 0} \sum_{\varphi(x)}^{f(x)} \Delta y \cdot \Delta x.
\end{aligned}
$$

The second integration amounts to finding the limit of the sum of such strips from $x = a$ to $x = b$ as $\Delta x \to 0$. Hence,

$$\int_a^b \left[\int_{\varphi(x)}^{f(x)} dy \right] dx = \lim_{\Delta x \to 0} \sum_a^b \left[\lim_{\Delta y \to 0} \sum_{\varphi(x)}^{f(x)} \Delta y \, \Delta x \right]$$

where the notation used on the right-hand side has a fairly obvious meaning.

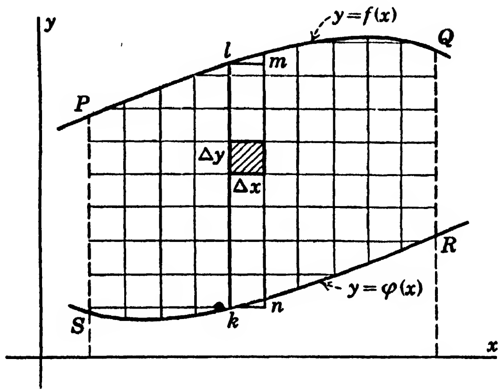

FIG. 153.

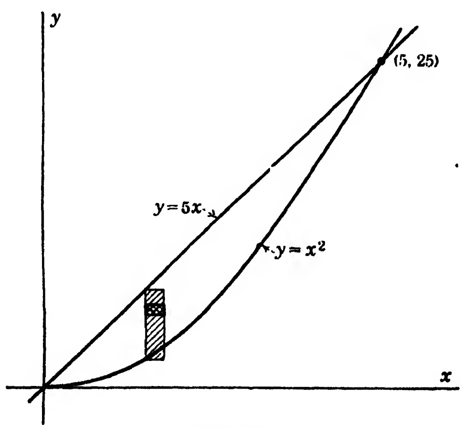

FIG. 154.

Example

Compute the area bounded by the parabola $y = x^2$ and the line $y = 5x$.

Solution (Fig. 154)

The curves intersect at (0, 0) and (5, 25); hence,

$$
\begin{aligned}
A &= \int_0^5 \int_{x^2}^{5x} dy\, dx \\
&= \int_0^5 y \Big]_{x^2}^{5x} dx \\
&= \int_0^5 (5x - x^2)dx \\
&= \tfrac{125}{6}.
\end{aligned}
$$

148. Volume by double integration.—We wish now to express as a double integral the volume between the

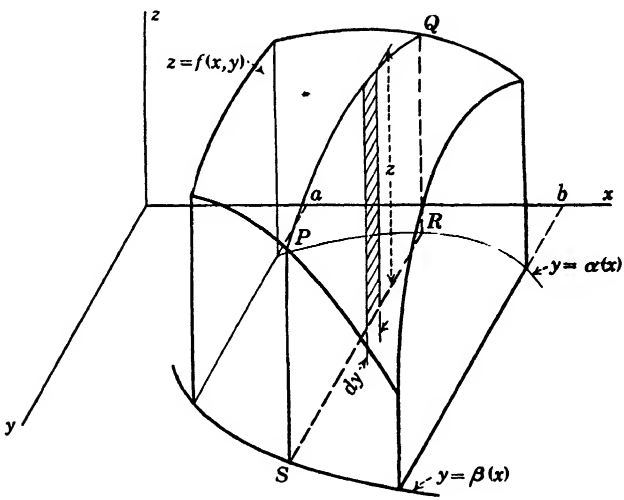

Fig. 155.

xy-plane and the surface $z = f(x, y)$, bounded on the sides by two arbitrary cylindrical surfaces $y = \alpha(x)$ and $y = \beta(x)$, and the planes $x = a$ and $x = b$ (Fig. 155).

From previous considerations we know that

$$V = \int_a^b A(x)dx$$

where $A(x)$ is the area of the section $PQRS$ at a distance x from the yz-plane. Also

$$A(x) = \int_{\alpha(x)}^{\beta(x)} z\, dy = \int_{\alpha(x)}^{\beta(x)} f(x, y)dy$$

where x is held constant during the integration. Hence,

$$V = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x, y)dy\, dx.$$

Example

Express the volume of one octant of the sphere $x^2 + y^2 + z^2 = r^2$, as a double integral.

Solution (Fig. 156)

$$\text{Area } PQR = A(x) = \int_0^{\sqrt{r^2-x^2}} z\, dy.$$

$$V = \int_0^r A(x)dx = \int_0^r \int_0^{\sqrt{r^2-x^2}} z\, dy\, dx$$

$$= \int_0^r \int_0^{\sqrt{r^2-x^2}} \sqrt{r^2 - x^2 - y^2}\, dy\, dx.$$

It is important to interpret this integration also as a double summation process. The quantity $z\,\Delta y\,\Delta x$ is the

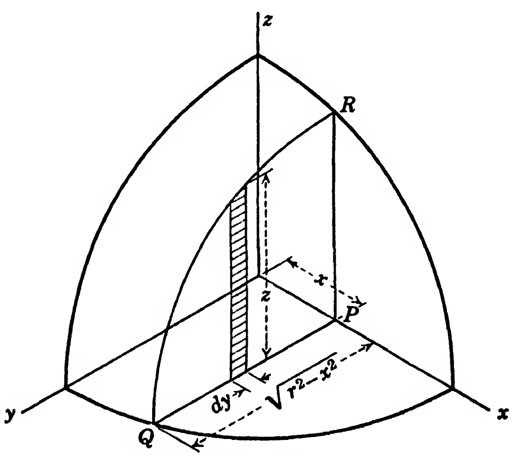

FIG. 156.

volume of a rectangular column with base Δy by Δx and height z, as indicated in Fig. 157. The student will easily see, using the figure, that the first integration (with respect to y) sums these columns into a slice extending from $y = 0$ to $y = \sqrt{r^2 - x^2}$. The second integration, then, sums all of the slices from $x = 0$ to $x = r$.

PROBLEMS

Using double integration, compute the area bounded by the given curve or curves.

1. $x^2 + y^2 = r^2$.

2. $b^2x^2 + a^2y^2 = a^2b^2$.

3. $y^2 = x^3$ and $y = x$.

4. $x = y^2 + 4y - 5$ and $x = 0$.

5. $y = 2x - x^2$ and $y = 3x^2 - 6x$.

6. $y^2 = x$ and $2y - x + 3 = 0$.

7. $y = \sin x$, $y = \cos x$, and $x = 0$.

8. $y = 5x$ and $y = x^3 + 4x^2$ (two areas).

9. Find the volume bounded by the coordinate planes and the plane $3x + 6y + 8z = 24$.

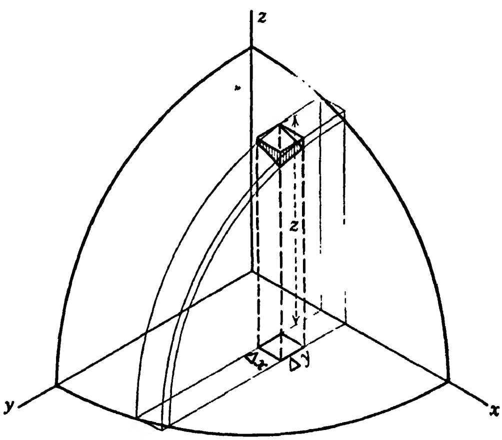

FIG. 157.

10. Find the volume bounded by the xy-plane, the paraboloid

$$x^2 + y^2 = 2z,$$

and the cylindrical surface $x^2 + y^2 = 4$.

11. The axes of two right circular cylinders of radius r intersect at right angles. Find the common volume.

12. Sketch the surface $x^2 + y^2 = 4 - z$. Set up a double integral for the volume bounded by it and the xy-plane.

13. Set up a double integral for the volume of the smaller of the two segments into which the sphere $x^2 + y^2 + z^2 = r^2$ is divided by the plane $z = \frac{1}{2}r$.

14. Set up a double integral for the volume above the xy-plane, bounded by the paraboloid $x^2 + y^2 = 8 - z$ and the cylinder

$$x^2 + y^2 = 4.$$

15. Set up a double integral for the volume common to the ellipsoid $\frac{x^2}{8} + \frac{y^2}{4} + \frac{z^2}{4} = 1$ and the cylinder $x^2 + y^2 = 4x$.

16. Set up a double integral for the volume bounded by the coordinate planes, the plane $x + y = 4$, and the paraboloid $z = x^2 + y^2$.

17. Set up a double integral for the volume of the wedge cut from the cylinder $x^2 + y^2 = 2x$ by the planes $z = 0$ and $z = x$.

18. Set up a double integral for the volume in the first octant under the plane $\frac{x}{6} + \frac{y}{4} + \frac{z}{8} = 1$ and inside the cylinder $x^2 + y^2 = 4$.

149. Moments. Product of inertia.—Suppose that one multiplies the area of each of the elements into which the area $PQRS$ is divided (Fig. 158) by the square of the distance of the element from the x-axis and adds together the quantities so obtained. The limit of this sum is evidently the second moment of the area with respect to the x-axis by a previous definition; *i.e.*,

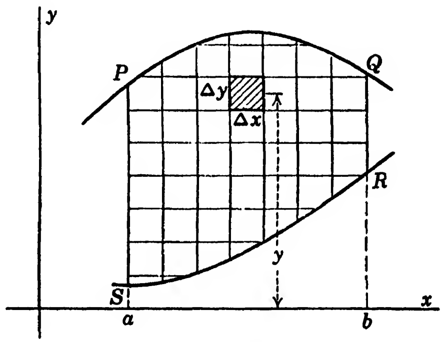

FIG. 158.

$$I_x = \iint y^2 dy\, dx$$

where the limits are to be chosen so as to cover the area under consideration. Similarly,

$$I_y = \iint x^2 dy\, dx$$
$$I_0 = \iint (x^2 + y^2) dy\, dx, \text{ etc.}$$

The value of

$$\iint xy\, dy\, dx$$

taken over an area is called the *product of inertia* of the

area with respect to the axes. Its value is, roughly speaking, the result of multiplying the area of each element by the product of its distances from the axes and adding the resulting quantities. It is important in connection with the theory of bending of beams and columns having unsymmetrical cross sections.

In general, the symbol

$$\iint_{(S)} f(x, y)dy\,dx$$

is called the integral of the function $f(x, y)$ over the area or region S. In Art. 148 we saw that such an integral may be interpreted as a volume We see now that it may have other interpretations,, depending upon the nature of the function $f(x, y)$ and on the physical or geometrical meanings of the variables themselves.

150. Volume by triple integration.—The student who has a clear mental picture of the process of computing area

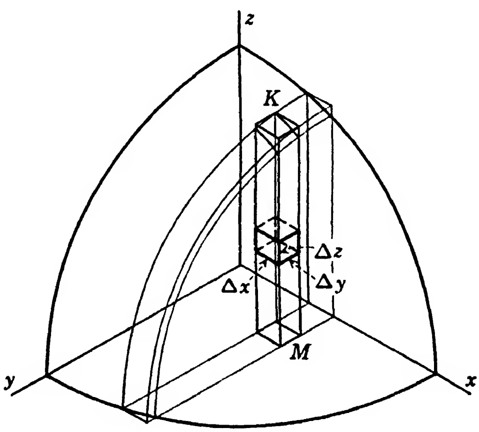

FIG. 159.

by means of a double integral will have no difficulty in extending his ideas to a third dimension. Just as an area is divided into rectangular elements by drawing lines parallel to the axes, a volume is divided into elements having the form of rectangular parallelepipeds (plus some little irregular pieces) by passing planes parallel to the

coordinate planes. If the octant of a sphere shown in Fig. 159 be so divided, the little parallelepiped with edges Δz, Δy, and Δx is a typical volume element. The whole volume is the limit of the sum of such elements. The summation may conveniently be carried out as follows:

1. Sum the elements in a vertical column extending from the xy-plane to the surface. This requires integration with respect to z from $z = 0$ to

$$z = f(x, y) = \sqrt{r^2 - x^2 - y^2}.$$

KM is a typical column.

2. Sum the columns in a slice parallel to the yz-plane extending from the x-axis to the bounding curve in the xy-plane. This requires integration with respect to y from $y = 0$ to $y = \varphi(x) = \sqrt{r^2 - x^2}$.

3. Sum the slices from $x = 0$ to $x = r$ by integrating with respect to x. We have, then,

$$V = \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\, dy\, dx.$$

151. Moments by triple integration.—Suppose that one multiplies the volume of each element in Fig. 159 by the square of the distance $(y^2 + z^2)$ of the element from the x-axis, and adds the resulting quantities. The limit of this sum is the value of I_x for the volume; *i.e.*,

$$I_x = \iiint (y^2 + z^2) dz\, dy\, dx$$

taken throughout the volume. Similarly,

$$I_{xz} = \iiint y^2 dz\, dy\, dx, \text{ etc.}$$

In general, the value of

$$\iiint f(x, y, z) dz\, dy\, dx$$

taken throughout a volume is called the integral of the function $f(x, y, z)$ over the volume. It may have various physical or geometrical interpretations depending upon the nature of the function $f(x, y, z)$ and on the physical meanings of the variables themselves.

PROBLEMS

1. Compute I_x for the area of the circle $x^2 + y^2 = r^2$, using both single and double integrals.

2. Show that $I_0 = I_x + I_y$.

3. Compute the product of inertia of the area of one quadrant of the circle $x^2 + y^2 = r^2$.

4. Compute the product of inertia of the area of a rectangle $ABCD$ with respect to sides AB and BC. $AB = 6$ in., $BC = 8$ in.

5. Evaluate the double integral of the function $f(x, y) = x^2 + y^2$ over the area of a circle with center at the origin and radius r. Interpret the result in two ways.

6. Evaluate $\int_0^2 \int_0^{x^2} xy\, dy\, dx$. Interpret the result.

7. Set up a triple integral for the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Draw a figure.

8. Using a triple integral, find the volume bounded by the cylinder $x^2 + y^2 = r^2$ and the planes $z = 0$ and $z = x$. Draw a figure.

9. Set up a triple integral for the volume bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

10. Set up a triple integral for the volume common to the cylinders $y^2 + z^2 = r^2$ and $x^2 + z^2 = r^2$.

11. Find the second moment of the mass of a solid right circular cylinder with respect to its axis, using two different integrals.

12. Compute the value of I for the cylinder of Prob. 11 with respect to an axis which coincides with a diameter of one end.

13. Sketch the surface $x^2 + 4y^2 = 32 + z$. Set up a triple integral for the volume bounded by it and the plane $z = 4$.

14. Set up a triple integral for the volume of the smaller of the two segments into which the sphere $x^2 + y^2 + z^2 = 16$ is divided by the plane $z = 2$.

15. Set up a triple integral for the volume in the first octant bounded by the planes $z = 0$ and $x + y + z = 4$, and the cylinder $x^2 + y^2 = 4$.

16. A solid cylinder with radius 6 in. and height 12 in. is cut by a plane which passes through a diameter of one base and is tangent to the other base. Find the volume of the smaller piece, using a triple integral.

152. Area in polar coordinates.—We wish now to express as a double integral in polar coordinates, the area bounded as indicated in Fig. 160 by two polar curves $\rho = f(\theta)$ and

$\rho = g(\theta)$, and the radial lines $\theta = \theta_1$ and $\theta = \theta_2$. In this case, instead of dividing the area into little rectangles by

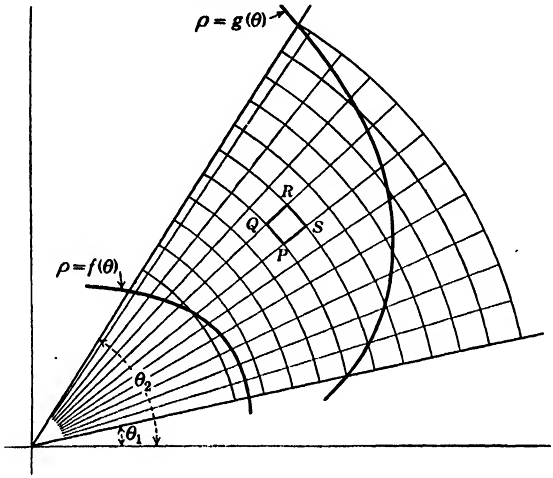

FIG. 160.

drawing lines parallel to the coordinate axes, we proceed as follows:

1. Draw a series of concentric circular arcs with radii differing by a small amount $\Delta\rho$; draw also a series of radial

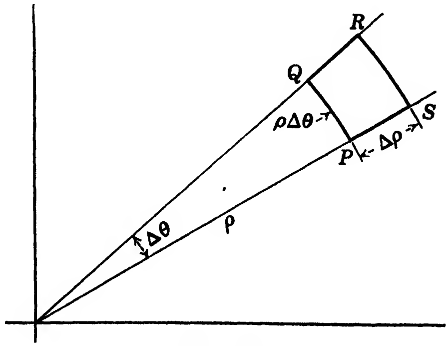

FIG. 161.

lines, the angle between any two being $\Delta\theta$. The required area is thus divided into little elements, one of which, *PQRS*, is shown to an enlarged scale in Fig. 161. There are of course some little irregular pieces left over.

2. The area of one element, found by taking the difference of two circular sectors, is

$$\begin{aligned}\Delta A &= \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta - \tfrac{1}{2}\rho^2\Delta\theta\\ &= \rho\,\Delta\rho\,\Delta\theta + \tfrac{1}{2}\,\Delta\rho^2\Delta\theta.\end{aligned}$$

3. The required area is clearly equal to the limit of the sum of the areas of these elements as $\Delta\rho$ and $\Delta\theta$ approach zero. In computing this limit we may, in accordance with Duhamel's theorem, drop the second term in the above expression for ΔA since it is of higher order than $\rho\,\Delta\rho\,\Delta\theta$, and proceed as though ΔA were equal to $\rho\,d\rho\,d\theta$. We have, then,

$$A = \int_{\theta_1}^{\theta_2}\int_{f(\theta)}^{g(\theta)} \rho\,d\rho\,d\theta.$$

The result is easily remembered by noting that the typical element is almost a rectangle with one side equal to $d\rho$ and the other equal to $\rho\,d\theta$.

153. Volume in cylindrical coordinates.—A coordinate system which is often useful is obtained by extending the two-dimensional system of polar coordinates to three dimensions. This is done by simply adding a third dimension z perpendicular to the plane in which ρ and θ are measured as indicated in Fig. 162. A point in space is specified by stating its three coordinates, ρ, θ, and z, the first two replacing the x and y of the rectangular coordinate system. When the equation of a surface is given in rectangular coordinates, it can be changed over into this new system of *cylindrical coordinates* by means of the relations

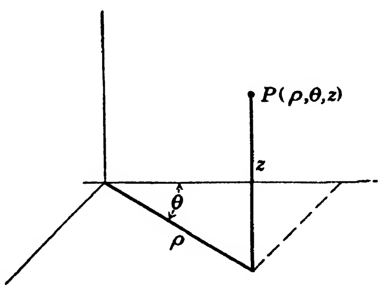

FIG. 162.

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta, \qquad z = z.$$

Thus the equation $x^2 + y^2 + z^2 = r^2$, for a sphere with

center at the origin and radius r, becomes in cylindrical coordinates,

$$\rho^2 + z^2 = r^2.$$

In order to express the volume of a solid as a triple integral in cylindrical coordinates we may, in an obvious way, divide it into elements having the form shown in

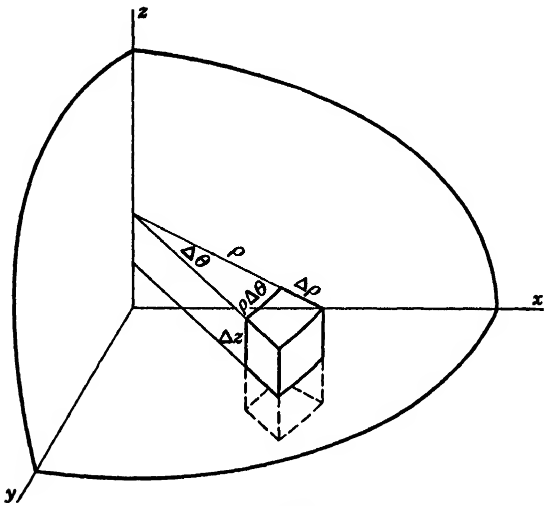

FIG. 163.

Fig. 163. The volume of such an element is, except for an infinitesimal of higher order,

$$\Delta v = \rho\, dz\, d\rho\, d\theta.$$

The volume of the solid is then given by

$$v = \iiint \rho\, dz\, d\rho\, d\theta;$$

the limits must of course be chosen so as properly to sum all of the elements.

Example

Find the volume of a sphere using cylindrical coordinates.

Solution (Fig. 164)

We may fix the limits so as to cover only the first octant and multiply by 8. Thus,

$$
\begin{aligned}
v &= 8\int_0^{\frac{\pi}{2}}\int_0^{r}\int_0^{\sqrt{r^2-\rho^2}} \rho\, dz\, d\rho\, d\theta \\
&= 8\int_0^{\frac{\pi}{2}}\int_0^{r} \sqrt{r^2-\rho^2}\,\rho\, d\rho\, d\theta \\
&= 8\int_0^{\frac{\pi}{2}} \frac{r^3}{3} d\theta \\
&= \frac{4\pi r^3}{3}.
\end{aligned}
$$

The student should see clearly that the first integration sums all of the elements in a vertical column extending

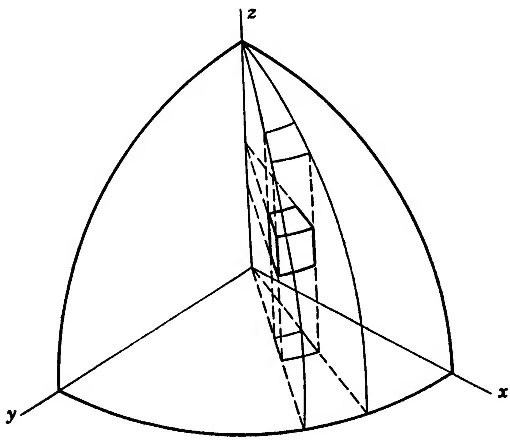

FIG. 164.

from the xy-plane to the surface; the second sums these columns into a wedge-shaped slice extending from the z-axis out to the circle $\rho = r$; the third sums up these slices.

PROBLEMS

Find each of the following areas using a double integral:

1. The area of the circle $\rho = 2r \sin \theta$.
2. The area bounded by the lemniscate $\rho^2 = a^2 \cos 2\theta$.
3. The area bounded by the cardioid $\rho = 1 + \sin \theta$.

4. The area inside the smaller loop of the curve $\rho = 2 \sin \theta + 1$.

5. The entire area bounded by the curve $\rho = a \sin 3\theta$.

6. The area enclosed by one loop of the curve $\rho = 2 \cos 2\theta$.

7. The area inside the circle $\rho = 8 \sin \theta$ and outside the circle $\rho = 4$.

8. The area inside the lemniscate $\rho^2 = 8 \cos 2\theta$ and outside the circle $\rho = 2$.

9. The area common to the circle $\rho = 3 \cos \theta$ and the cardioid $\rho = 1 + \cos \theta$.

10. Find the volume of one octant of the sphere $\rho^2 + z^2 = r^2$ by integrating first with respect to z, then θ, then ρ. Draw a corresponding figure showing how the elements are summed up.

11. Find the volume of a right circular cylinder of radius r and height h using cylindrical coordinates. Draw the figure.

12. Find the volume bounded by the xy-plane and the paraboloid $x^2 + y^2 = 4 - z$.

13. Find the volume above the xy-plane which is inside the cylinder $x^2 + y^2 = 4$ and under the paraboloid $x^2 + y^2 = z - 2$.

14. Find the volume above the xy-plane which is inside the cylinder $x^2 + y^2 = 4$ and under the cone $x^2 + y^2 = z^2$.

15. Find the volume bounded by the cylinder $\rho = 2r \cos \theta$ and the planes $z = 0$ and $z = \rho \cos \theta$.

16. Find the volume common to the sphere $x^2 + y^2 + z^2 = 25$ and the cylinder $x^2 + y^2 = 9$.

17. Find the volume common to the sphere $x^2 + y^2 + z^2 = 16$ and the cylinder $\rho = 4 \cos \theta$.

18. Find the volume above the xy-plane which is under the cone $x^2 + y^2 = z^2$ and inside the cylinder $x^2 + y^2 = 4x$.

19. Find the volume above the xy-plane which is under the paraboloid $x^2 + y^2 = z$ and inside the cylinder $x^2 + y^2 = 4x$.

20. Find the volume bounded by the cone $x^2 + y^2 = z^2$ and the paraboloid $x^2 + y^2 = 8z - 16$.

CHAPTER XXVI

INFINITE SERIES

154. Introduction.—The student has already encountered infinite series in a variety of ways. Using the binomial theorem for negative or fractional exponents leads, for example, to such a series. Thus

$$(1 + x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \cdots,$$

the dots indicating that the terms continue indefinitely. This series can be used to calculate $\sqrt{1 + x}$ for values of x numerically less than 1. Thus, taking $x = \frac{1}{4}$, we get

$$\sqrt{5} = 2\left[1 + \frac{1}{2(4)} - \frac{1}{8(16)} + \frac{1}{16(64)} - \cdots\right]$$

from which, by taking enough terms, one can compute $\sqrt{5}$ to any number of decimal places.

As another example, by ordinary long division one finds

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots.$$

If we integrate both sides between the limits $x = 0$ and $x = x$, we get

$$\text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

If this integration is permissible and if the result is valid for $x = 1$, we should have

$$\text{arc tan } 1 = \frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots,$$

or

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots)$$

from which π could be computed to any desired degree of accuracy.*

We must obviously be able to determine under what conditions and for what values of x one can use an infinite series in the manner just indicated. These questions will be considered in this and the following chapter.

155. Definitions.—A succession of numbers or terms formed according to a definite rule is called a **sequence.** Thus, the numbers 1, 4, 9, 16, form a sequence. The indicated sum of the terms of a sequence is called a **series;** it is a *finite* or an *infinite* series according as the number of terms is limited or unlimited. Thus, the indicated sum

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

is an infinite series.

By inspecting carefully the first few terms, one can usually discover the rule by which the successive terms are formed and write down a formula for the general or nth term.

Examples

Series	nth term
$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$	$\dfrac{1}{n^2}$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$	$\dfrac{1}{2^{n-1}}$
$x^2 - x^4 + x^6 - x^8 + \cdots$	$(-1)^{n-1}x^{2n}$

156. Sum of an infinite series.—By the sum of a *finite* series we mean of course the algebraic sum of all the terms. In this sense an *infinite* series has no sum; for, no matter how many terms we might add up, there would always be an indefinite number of them left over. We shall, however, attach a meaning to the word *sum* as applied to an infinite series by the following definition:

Consider a series

$$u_1 + u_2 + u_3 + u_4 + \cdots$$

* Fewer terms would be required if one made use of the relation arc tan 1 = arc tan $\frac{1}{4}$ + arc tan $\frac{3}{5}$.

and let S_n denote the sum of the first n terms; i.e., let

$$
\begin{aligned}
S_1 &= u_1 \\
S_2 &= u_1 + u_2 \\
S_3 &= u_1 + u_2 + u_3 \\
&\cdots\cdots \\
S_n &= u_1 + u_2 + u_3 + \cdots + u_n.
\end{aligned}
$$

If S_n, regarded as a function of n, approaches a limit S as $n \to \infty$, this limit S is called the **sum** *of the infinite series and the series is said to be* **convergent.** *If S_n does not approach a limit as $n \to \infty$ the series is said to be* **divergent.**

Example 1

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots.$$

In this case,

$$
\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \tfrac{1}{2} = 1\tfrac{1}{2} \\
S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} = 1\tfrac{3}{4} \\
S_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = 1\tfrac{7}{8} \\
&\cdots\cdots
\end{aligned}
$$

It is *almost* obvious that $\lim\limits_{n\to\infty} S_n = 2$. That this is actually true is easily shown in this case, because the terms of this particular series form a *geometric progression;* from the formula for the sum of n terms of such a progression, we have

$$S_n = 2[1 - (\tfrac{1}{2})^n].$$

From this expression we see easily that

$$\lim_{n\to\infty} S_n = 2;$$

i.e., the series is convergent with sum 2.

Example 2

$$1 + 2 + 3 + 4 + \cdots.$$

In this case, $S_1 = 1$, $S_2 = 3$, $S_3 = 6$, etc. Obviously $S_n \to \infty$ as $n \to \infty$; *i.e.*, the series is divergent.

Example 3

$$1 - 1 + 1 - 1 + \cdots.$$

In this case, $S_1 = 1$, $S_2 = 0$, $S_3 = 1$, $S_4 = 0$, etc. Certainly S_n does not approach a limit as $n \to \infty$. The series is therefore divergent. It is

sometimes called *finitely oscillating* to distinguish its behavior from that of the series in Example 2.

The convergence or divergence of a series cannot usually be established by studying directly the behavior of S_n, for ordinarily one cannot express S_n in terms of n. Consider, for example, the *harmonic series*,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots .$$

Here, $S_1 = 1$, $S_2 = 1\tfrac{1}{2}$, $S_3 = 1\tfrac{5}{6}$, etc. Whether or not S_n approaches a limit as $n \to \infty$ is not immediately apparent; furthermore, we cannot see any way of expressing S_n as a simple function of n as was done in Example 1 above. The convergence or divergence must then be determined by a study of the series itself. It will be seen later that the series is divergent; *i.e.*, S_n increases indefinitely with increasing n.

157. A necessary condition for convergence.—It seems fairly obvious that a series cannot converge unless the terms approach zero as we go farther and farther out in the series; for, unless this were true, each term added would change S_n by an amount not approaching zero and S_n could not approach a limit. It is *necessary* then for convergence of the series $u_1 + u_2 + u_3 + \cdots$ that

$$\lim_{n\to\infty} u_n = 0.$$

This condition is *not* however sufficient; *i.e.*, a series may be divergent even when the condition is satisfied. Thus it has already been mentioned (without proof) that the harmonic series $1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$ is divergent, and it is obvious that $u_n \to 0$ as $n \to \infty$.

PROBLEMS

Assuming the law of formation suggested by the given terms, write down a formula for the nth term.

1. $1 + 3 + 5 + 7 + \cdots .$

2. $1 + \dfrac{2}{2} + \dfrac{3}{2^2} + \dfrac{4}{2^3} + \cdots .$

3. $\tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{10} + \tfrac{1}{17} + \tfrac{1}{26} + \cdots .$

4. $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} - \frac{1}{7 \cdot 8} + \cdots.$

5. $\frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \cdots.$

6. $\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \cdots.$

7. $\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} - \frac{4}{\sqrt{17}} + \cdots.$

8. $\frac{1 \cdot 2}{3^3} + \frac{2 \cdot 3}{4^3} + \frac{3 \cdot 4}{5^3} + \cdots.$

9. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots.$

In each of the following, the general term is given; write down the first four terms.

10. $\frac{3^n}{\sqrt{n}}.$

11. $(-1)^n \frac{n}{n^2 + 1}.$

12. $\frac{x^n}{(n + 1)!}.$

13. $(-1)^{n+1} \frac{n + 1}{n \log (n + 1)}.$

14. $\frac{n!}{2^{n-1}}.$

15. $(-1)^{n-1} \frac{2^{2n-1}}{n!}.$

16. Show that the geometric series $a + ar + ar^2 + \cdots$ is convergent with sum $\frac{a}{1 - r}$ if $|r| < 1$, and divergent if $|r| \geq 1$.

17. Write out the binomial expansion of $(1 + x)^{\frac{1}{2}}$ and substitute $x = \frac{1}{9}$. Under what condition could the resulting series be used to compute $\sqrt{10}$?

Prove that the following series are divergent:

18. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots.$

19. $\frac{2}{\log 2} + \frac{3}{\log 3} + \frac{4}{\log 4} + \cdots.$

20. $\frac{2}{3} - \frac{3}{6} + \frac{4}{9} - \frac{5}{12} + \cdots.$

21. $\frac{3}{3} + \frac{5}{6} + \frac{7}{9} + \frac{9}{12} + \frac{11}{15} + \cdots.$

22. Prove that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

HINT:

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$
$$\frac{1}{9} + \cdots + \frac{1}{16} > \frac{1}{16} + \cdots + \frac{1}{16} = \frac{1}{2}, \text{ etc.}$$

23. Explain how the idea of the "sum" of an infinite series as here defined is related to that of the limit of a function as defined in Chap. II.

158. Tests for convergence.—We have seen that the first step in examining a given series for convergence is to determine whether or not $u_n \to 0$ as $n \to \infty$. If $\lim_{n \to \infty} u_n \neq 0$ the series diverges. If the limit *is* zero the series may either converge or diverge, and further tests must be applied.

There is no general test which will serve in all cases; instead, we have a multitude of special tests, any one of which may serve in a particular case. Several of the most important of these are given in the next few pages.

In the proofs of some of the tests we shall use the following obvious

Fundamental principle: *If a variable quantity steadily increases but never becomes greater than a fixed quantity Q, it must approach a limit which is not greater than Q.*

159. The integral test.—Suppose that the individual terms of the series

$$u_1 + u_2 + u_3 + \cdots$$

are all *positive* and decrease with increasing n, approaching zero as $n \to \infty$.

By erecting ordinates equal to $u_1, u_2, u_3, \cdots$ at the points $x = 1, x = 2, x = 3, \cdots$, the magnitudes of the terms may be represented by the *areas* of the rectangles shown in Fig. 165.

Consider now a function $f(x)$ where $f(n)$ is the general or nth term of the series. Its graph passes through the upper right-hand corner of each rectangle; furthermore, if $f(x)$ is positive, continuous, and everywhere decreasing in the interval $0 \leq x < \infty$, the general appearance of the graph will be as indicated in the figure.

It is evident that S_n (which is represented by the sum of the areas of the first n rectangles) is a quantity which always increases with n, but remains less than the area under the curve in the interval from $x = 0$ to $x = n$. Now, if the area under the curve remains finite as $x \to \infty$,

it follows from our fundamental principle of the last article that S_n must approach a limit as $n \to \infty$; *i.e.*,

$$\lim_{n \to \infty} S_n = S \text{ exists, if } \int_0^\infty f(x)dx \text{ exists.}$$

It can easily be shown that if the improper integral does not exist the series is divergent. The proof is left to the student, Fig. 166 providing a sufficient hint.

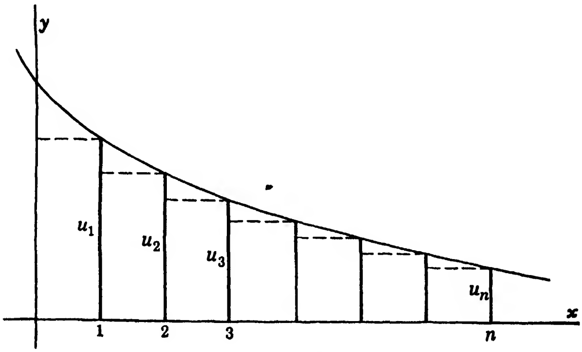

FIG. 165.

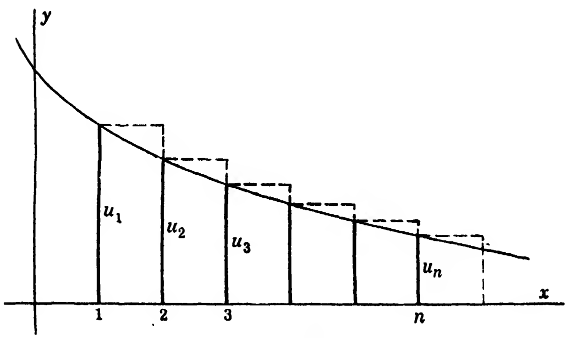

FIG. 166.

It is important to remark here that the convergence or divergence of a series is not affected by adding or dropping any *finite* number of terms at the beginning of the series. This means, in the first place, that in applying any test we do not have to use exactly the nth term. The $(n + 1)$th term or *any* general term will serve just as well. For

example, in testing the series

$$\tfrac{1}{5} + \tfrac{1}{10} + \tfrac{1}{17} + \tfrac{1}{26} + \cdots$$

we may take $\dfrac{1}{n^2 + 1}$ as the general term even though the nth term is really $\dfrac{1}{(n+1)^2 + 1}$. It is easier to use $\dfrac{1}{n^2 + 1}$ and the change amounts to adding the term $\frac{1}{2}$ at the beginning of the series.

Another consequence of the above remark is that the conditions imposed above upon the function $f(x)$ need not be satisfied for *all* positive values of x, but only for all values from a certain point on—say for all $x > a$. Any irregularity in the initial part can be avoided by starting farther out in the series. Finally, then, we may state the integral test as follows:

Suppose that $f(x)$ is a function such that $f(n)$ is the general term of the series $u_1 + u_2 + u_3 + \cdots$; suppose furthermore that for all $x > a$, $f(x)$ is defined, is positive, and decreases with increasing x; then, the series converges or diverges according as $\int_a^\infty f(x)dx$ does or does not exist.

Example 1

$$\tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{10} + \tfrac{1}{17} + \cdots.$$

The general term is $\dfrac{1}{n^2 + 1}$; the function $f(x)$ is $\dfrac{1}{x^2 + 1}$; this function is defined for all $x \geqq 0$ and is everywhere positive and decreasing. Also,

$$\int_0^\infty \frac{dx}{x^2 + 1} = \text{arc tan } x \Big]_0^\infty {}^* = \frac{\pi}{2};$$

hence the series converges. Its sum, which can be approximated by adding several terms is clearly less than $\pi/2$.

Example 2

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots.$$

The general term is $1/n$; the function $f(x)$ is $1/x$; this function is undefined at $x = 0$ but starting say at $x = 1$ all necessary conditions are satisfied. Also,

* This of course means $\lim\limits_{h \to \infty} \Big[\text{arc tan } x\Big]_0^h$. See pp. 199–200.

$$\int_1^\infty \frac{dx}{x} = \log x \Big]_1^\infty = \infty;$$

since the integral does not exist, the series is divergent.

PROBLEMS

1. Show that the "k-series" $1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$ is convergent if $k > 1$, divergent if $k \leqq 1$.

2. Show that the geometric series $a + ar + ar^2 + ar^3 + \cdots$ is convergent if $|r| < 1$, divergent if $|r| \geqq 1$.

Test the following series:

3. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots.$

4. $1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots.$

5. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots.$

6. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \cdots.$

7. $\frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \cdots.$

8. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$

9. $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$

10. $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} + \frac{4}{\sqrt{17}} + \cdots.$

11. $\frac{1}{2} + \frac{4}{2^3 + 1} + \frac{9}{3^3 + 1} + \frac{16}{4^3 + 1} + \cdots.$

12. $\frac{1}{101} + \frac{2}{102} + \frac{3}{103} + \cdots.$

13. $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \cdots.$

14. $\frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 4}{4} + \cdots.$

15. In stating the integral test we have specified that $f(x)$ should be a decreasing function. Is it sufficient to specify that $f(x)$ should *never increase* with x (for $x > a$)? Illustrate your answer with a graph.

16. Prove the integral test for the case of divergence.

160. The comparison test.—One can often prove the convergence or divergence of a given series by comparing it with a series whose behavior is known. In this connection we have the following two theorems:

1. *A series of positive terms is* **convergent** *if each of its terms is less than (or equal to) the corresponding term of a series which is known to be convergent.*

2. *A series of positive terms is* **divergent** *if each of its terms is greater than (or equal to) the corresponding term of a series which is known to be divergent.*

The student will probably accept these theorems as obvious; formal proofs are left to the exercises.

In order to use the comparison test one must have in mind several series which are known to be convergent and several which are divergent. Thus, we know now that the harmonic series is divergent. We know also the behavior of the "k-series" and the geometric series (Probs. 1 and 2 of the last set).

Example 1

$$1+\frac{1}{\log 2}+\frac{1}{\log 3}+\frac{1}{\log 4}+\cdots.$$

Each term (after the first) is larger than the corresponding term of the harmonic series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots$$

which is known to be divergent. Hence, the given series is divergent.

Example 2

$$\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{9}+\tfrac{1}{17}+\cdots+\frac{1}{2^n+1}+\cdots.$$

Each term is smaller than the corresponding term of the geometric series

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\cdots$$

which is known to be convergent. Hence the given series is convergent.

161. The ratio test.—The tests which have just been given are essentially for series with *positive* terms. One of the most useful tests, and one which can be applied to *any* series not merely to series of positive terms, is the following:

Ratio test: *Given the series*

$$u_1+u_2+u_3+\cdots+u_n+u_{n+1}+\cdots;$$

form the ratio u_{n+1}/u_n of a general term to the term preceding it. Then

(1) If $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right|<1$, *the series converges.*

(2) If $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right|>1$, *or if* $\dfrac{u_{n+1}}{u_n}\to\infty$ *as* $n\to\infty$, *the series diverges.*

(3) $$\text{If } \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1, \textit{ the test fails.}$$

Example

Test the series $1 - \frac{2}{3} + \frac{3}{3^2} - \frac{4}{3^3} + \cdots$.

Solution

Here, apart from the sign which may be disregarded,

$$u_n = \frac{n}{3^{n-1}} \qquad \text{and} \qquad u_{n+1} = \frac{n+1}{3^n};$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{n+1}{3^n}\cdot\frac{3^{n-1}}{n} = \frac{n+1}{3n}.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{n+1}{3n} = \frac{1}{3}.$$

Since the limit is *less* than 1 the series is convergent.

PROOF OF TEST: We shall give the proof of part (1) for the case in which all terms are positive. If

$$\lim \frac{u_{n+1}}{u_n} = k < 1,$$

then for all sufficiently large n the value of the ratio u_{n+1}/u_n is less than some proper fraction r which is itself less than 1; *i.e.*, for $n > n'$

$$\frac{u_{n+1}}{u_n} < r, \qquad \text{or} \qquad u_{n+1} < u_n r$$

$$\frac{u_{n+2}}{u_{n+1}} < r, \qquad \text{or} \qquad u_{n+2} < u_n r^2$$

$$\frac{u_{n+3}}{u_{n+2}} < r, \qquad \text{or} \qquad u_{n+3} < u_n r^3$$

. . . .

We see immediately that, discarding the first n' terms, the remaining ones are smaller than the corresponding terms of the *convergent* geometric series

$$u_n r + u_n r^2 + u_n r^3 + \cdots \quad (0 < r < 1).$$

Hence, the series is convergent by the comparison test.

The proof of part (2) is obtained in a similar manner, it being easy to show that in this case the terms eventually increase and $\lim_{n\to\infty} u_n \neq 0$.

The proof of part (3) is obtained by finding both a convergent and a divergent series for which $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. It can easily be shown, for example, that in the case of the series

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots,$$

the value of this limit is 1 for all finite values of k. The series, however, is convergent if $k > 1$ and divergent if $k < 1$.

162. The polynomial test.*—Because of the frequent occurrence of series in which the general term u_n is the quotient of two polynomials, the following test is often useful.

Polynomial test: *Let* $|u_n| = p_r(n)/p_s(n)$ *where* $p_r(n)$ *and* $p_s(n)$ *are polynomials in* n *of degree* r *and* s*, respectively. Then the series is convergent if* $s - r > 1$ *and divergent if* $s - r < 1$*. If* $s - r = 1$ *the test gives no information unless all the* u_n *have the same sign, in which case the series is divergent.*

Example 1

$$1 + \tfrac{3}{8} + \tfrac{5}{27} + \tfrac{7}{64} + \cdots.$$

Here $u_n = \dfrac{2n - 1}{n^3}$; the degree of the denominator exceeds that of the numerator by 2; the series is therefore convergent. A method of proving the test immediately suggests itself. For we may factor out n^2 in the denominator and write the general term in the form $u_n = \left(\dfrac{2n-1}{n}\right)\left(\dfrac{1}{n^2}\right)$; then, since the limit of the first factor as $n \to \infty$ is a constant, the terms of the series, at least from a certain point on, are smaller than those of a series whose general term is $k(1/n^2)$ where k is a sufficiently large constant.

* The name appears to be due to Professor C. A. Hutchinson of the University of Colorado.

Example 2

$$\tfrac{3}{2} + \tfrac{4}{5} + \tfrac{5}{10} + \tfrac{6}{17} + \cdots .$$

Here $u_n = \dfrac{n+2}{n^2+1}$; the degree of the denominator exceeds that of the numerator by only 1; the series is, therefore, *divergent*. A method of proving this part of the test is also evident. For we may factor out n in the denominator and write u_n in the form $u_n = \left(\dfrac{n+2}{n+\dfrac{1}{n}}\right)\left(\dfrac{1}{n}\right)$; then, since the limit of the first factor is a constant, the terms are larger than those of a series whose general term has the form $k(1/n)$ where k is a sufficiently small constant.

PROBLEMS

1. Write out a proof of the comparison test for the case of (a) convergence, (b) divergence.

2. Write out a proof of part (2) of the ratio test assuming positive terms.

Test the following series:

3. $\dfrac{\sqrt{1}}{2} + \dfrac{\sqrt{2}}{3} + \dfrac{\sqrt{3}}{4} + \cdots .$ **4.** $1 + \dfrac{1}{2^2} + \dfrac{1}{3^3} + \dfrac{1}{4^4} + \cdots .$

5. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots .$ **6.** $\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \cdots .$

7. $\tfrac{1}{1000} + \tfrac{1}{2000} + \tfrac{1}{3000} + \cdots .$

8. $1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots .$

9. $\dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \cdots .$

10. $\dfrac{1!}{10} + \dfrac{2!}{10^2} + \dfrac{3!}{10^3} + \cdots .$ **11.** $\dfrac{3}{1^3} + \dfrac{6}{2^3} + \dfrac{9}{3^3} + \cdots .$

12. $\dfrac{1\cdot 2}{3^3} + \dfrac{2\cdot 3}{4^3} + \dfrac{3\cdot 4}{5^3} + \cdots .$

13. $\dfrac{2}{1} + \dfrac{2^2}{2} + \dfrac{2^3}{3} + \cdots .$ **14.** $1 + \dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \cdots .$

15. $1 - \dfrac{1}{2} + \dfrac{2}{2^2} - \dfrac{3}{2^3} + \cdots .$

16. $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{3\cdot 4\cdot 5} + \dfrac{1}{5\cdot 6\cdot 7} + \cdots .$

17. $\frac{2}{1!} - \frac{3}{2!} + \frac{4}{3!} - \frac{5}{4!} + \cdots$. **18.** $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$.

19. $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$. **20.** $1 - \frac{2}{3!} + \frac{3}{5!} - \frac{4}{7!} + \cdots$.

21. $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \cdots$.

163. Alternating series.—A series is said to be *alternating* if the terms are alternately positive and negative. For convergence of such a series it is sufficient that

1. *Each term be numerically less than the preceding one, and*
2. *The nth term approach zero as a limit as* $n \to \infty$.

PROOF: Consider the series

$$u_1 - u_2 + u_3 - \cdots \pm u_n \mp u_{n+1} \pm \cdots$$

where

$$u_1 > u_2 > u_3 \cdots > 0 \qquad \text{and} \qquad \lim_{n\to\infty} u_n = 0.$$

The sum S_{2n} of an *even* number of terms can be written in either of the two following ways:

$$(1)\ S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots + (u_{2n-1} - u_{2n}).$$

$$(2)\ S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n}.$$

Form (1) shows that S_{2n} is a positive increasing function of n while form (2) shows that its value remains *less* than u_1. Hence S_{2n} must approach a limit as $n \to \infty$. The sum S_{2n+1} of an odd number of terms is

$$S_{2n+1} = S_{2n} + u_{2n+1}.$$

Since $u_{2n+1} \to 0$, it is clear that S_{2n+1} approaches the same limit as S_{2n} when $n \to \infty$. The series is then convergent.

It is impossible in general to find exactly the sum of a convergent series. It follows from the definition of the sum, however, that an arbitrarily close approximation can be obtained by adding up a sufficient number of terms at the beginning of the series. The degree of accuracy with

which such an approximation represents the sum of the series is in general difficult to estimate. However, in the case of an *alternating series*, this question is answered by the following:

Theorem: *The error involved in approximating the sum of a convergent alternating series (of the type here discussed) by adding the first n terms, is less numerically than the first neglected term.*

The proof is left to the exercises.

164. Series of mixed terms—absolute convergence.—A series which contains an unlimited number of both positive and negative terms (not necessarily alternating) is called a *mixed-term* series. It can be shown rather easily that such a series is always convergent *provided the series obtained from it by making all terms positive is convergent.* In this case the original series is said to be **absolutely** convergent.

The tests already given for positive term series may of course be used to test a series of mixed terms for absolute convergence.

Example

The series

$$1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \cdots$$

in which each positive term is followed by two negative terms is absolutely convergent since the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent series.

A series of mixed terms *may* be convergent when the series obtained by making all terms positive is *divergent.* In this case the original series is said to be **conditionally** convergent.

Example

The series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots$$

which is convergent by the alternating series test is only *conditionally* convergent, since the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots$$

is divergent.

The conceptions of absolute and conditional convergence are necessary in the study of the algebra of convergent series. Thus, if one multiplies together two convergent series, the product series is not necessarily convergent unless at least one of the original series is absolutely convergent. It can be shown also that the sum of an absolutely convergent series is not affected by a rearrangement of the terms, while a change in the order of the terms of a conditionally convergent series may lead to an entirely different sum or even to a divergent series. Thus, the series

$$1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \cdots$$

is merely a rearrangement of the conditionally convergent series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots .$$

It can be shown that the first series is convergent and its sum is half that of the second series.

PROBLEMS

Show that the following series are convergent. Determine in each case whether the convergence is absolute or conditional.

1. $\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots .$

2. $\dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \dfrac{1}{\log 5} + \cdots .$

3. $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \cdots .$

4. $\dfrac{1}{2!} - \dfrac{2}{3!} + \dfrac{3}{4!} - \dfrac{4}{5!} + \cdots .$

5. $1 - \dfrac{1}{2^2} + \dfrac{1}{4^2} - \dfrac{1}{6^2} + \cdots .$

6. $\dfrac{1}{2^2} - \dfrac{2}{2^3} + \dfrac{3}{2^4} - \dfrac{4}{2^5} + \cdots .$

7. $\dfrac{2^2}{1^3} - \dfrac{4^2}{2^3} + \dfrac{6^2}{3^3} - \dfrac{8^2}{4^3} + \cdots .$

8. $\dfrac{2^2}{4^0} - \dfrac{4^2}{4^1} + \dfrac{6^2}{4^2} - \dfrac{8^2}{4^3} + \cdots .$

9. $\tfrac{1}{2} - \tfrac{2}{5} + \tfrac{3}{10} - \tfrac{4}{17} + \cdots .$

10. $\frac{1}{3} - \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots .$

11. $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots .$

12. Expand $(x + 1)^{1/3}$ by the binomial theorem. From the series compute $\sqrt[3]{28}$ approximately using three terms. Estimate the maximum error involved.

13. Compute π to four decimal places from the series given in Art. 154. Make use of the footnote.

14. Show that the series

$$\tfrac{1}{2} - \tfrac{3}{5} + \tfrac{5}{10} - \tfrac{7}{17} + \cdots$$

is conditionally convergent. How many terms would be required to insure accuracy to two decimal places in computing its sum?

15. Approximate the sum of the series

$$\frac{1}{4} + \frac{1}{2 \cdot 4^2} + \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} + \frac{1}{5 \cdot 4^5} + \cdots$$

by adding up the first four terms. Try to estimate the error.

HINT:

$$\begin{aligned}\frac{1}{5 \cdot 4^5} + \frac{1}{6 \cdot 4^6} + \frac{1}{7 \cdot 4^7} + \cdots &< \frac{1}{5 \cdot 4^5} + \frac{1}{5 \cdot 4^6} + \frac{1}{5 \cdot 4^7} + \cdots \\ &= \frac{1}{5}\left(\frac{1}{4^5} + \frac{1}{4^6} + \frac{1}{4^7} + \cdots\right) \\ &= \frac{1}{5}\left(\frac{\frac{1}{4^5}}{1 - \frac{1}{4}}\right) = \frac{4}{3}\left(\frac{1}{5 \cdot 4^5}\right).\end{aligned}$$

The error is then less than $\frac{4}{3}$ the first neglected term.

16. Write out a proof of the theorem concerning the error involved in approximating the sum of a convergent alternating series.

165. Series of variable terms.—We have been considering series in which the individual terms were constants. A more general case is that in which the terms are functions of some independent variable x. Thus, the following are all *variable-term* series:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots . \tag{1}$$

$$(x - 2) + 2(x - 2)^2 + 3(x - 2)^3 + \cdots . \tag{2}$$

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots . \tag{3}$$

The first is called a *power series* in x; the second, a power series in $(x - 2)$. The third series is called a trigonometric series. In each case the name is descriptive of the way in which x occurs in the series.

In general, series like the above are convergent for some values of x and divergent for others; in particular, a power series is convergent for all values of x in a certain interval, called the *interval of convergence*, and divergent for all values of x outside this interval. Since power series are of great importance, we shall consider next the problem of determining this interval.

166. Interval of convergence of a power series.—A power series in x is any series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

where the a's are constants. An interval such that the series converges for all values of x inside and diverges for all values outside can always be found by using the ratio test. The behavior for the two values of x representing the end points of this interval must be determined by substituting these values for x in the series and testing the resulting series of constant terms.

Example

Find the interval of convergence of the series

$$\frac{x}{1 \cdot 3} + \frac{x^2}{2 \cdot 3^2} + \frac{x^3}{3 \cdot 3^3} + \frac{x^4}{4 \cdot 3^4} + \cdots.$$

Solution

The ratio of a general term to the preceding one is

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)3^{n+1}}\,\frac{n \cdot 3^n}{x^n} = \frac{nx}{3(n+1)}.$$

The limit approached by this ratio as $n \to \infty$ is

$$\lim_{n \to \infty} \frac{nx}{3(n+1)} = \frac{x}{3}.$$

The series is convergent for all values of x for which the absolute value of this limit is *less than* 1; *i.e.*, for

$$\left|\frac{x}{3}\right| < 1, \qquad \text{or} \qquad |x| < 3, \qquad \text{or} \qquad -3 < x < 3.$$

The series is divergent for all values of x for which this limit is *more than* 1 *in absolute value*, *i.e.*, for x more than 3 or less than -3.

To test the end points of the interval, we substitute the values 3 and -3 for x in the given series obtaining

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots \text{ if } x = 3,$$
$$-1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \cdots \text{ if } x = -3.$$

The first is the divergent harmonic series but the second is convergent by the alternating series test. The complete interval of convergence is then expressed by

$$-3 \leqq x < 3$$

and is represented graphically by the heavy line in Fig. 167. The small circle drawn about the point $x = 3$ indicates that this point is not included in the interval.

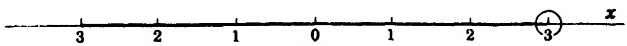

FIG. 167.

That the interval of convergence of any power series in x is bisected by the origin is evident from the manner in which the interval is determined; *i.e.*, the above process would give, in general, an interval from $x = -k$ to $x = +k$ including neither or one or both of the end points. This interval may be so small in some cases as to reduce to the single point $x = 0$ $\left(\text{if } \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \infty\right)$, or it may be so large as to include all values of x $\left(\text{if } \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} \equiv 0\right)$.

Any series of the form

$$a_0 + a_1(x - b) + a_2(x - b)^2 + \cdots$$

is called a power series in $x - b$. The procedure for finding its interval of convergence is the same as that used for a power series in x.

Example

Find the interval of convergence of the series

$$\frac{x-3}{1 \cdot 2} + \frac{(x-3)^2}{3 \cdot 2^2} + \frac{(x-3)^3}{5 \cdot 2^3} + \frac{(x-3)^4}{7 \cdot 2^4} + \cdots.$$

Solution

Using the ratio test, we find that

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(2n+1)\cdot 2^{n+1}}\cdot\frac{(2n-1)\cdot 2^n}{(x-3)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x-3}{2}\cdot\frac{2n-1}{2n+1}\right| = \left|\frac{x-3}{2}\right|.$$

The series is convergent, if

$$\left|\frac{x-3}{2}\right| < 1, \qquad \text{or} \qquad |x-3| < 2, \qquad \text{or} \qquad 1 < x < 5.$$

It is easy to show that the series converges if $x = 1$ and diverges if $x = 5$; the complete interval of convergence is then expressed by

$$1 \leqq x < 5$$

and is represented graphically in Fig. 168. It should be noticed that

FIG. 168.

the interval of convergence of a power series in $x - b$ is bisected by the point $x = b$.

PROBLEMS

Find the interval of convergence of the following power series:

1. $x + 2x^2 + 3x^3 + \cdots.$

2. $x + \dfrac{x^3}{\sqrt{3}} + \dfrac{x^5}{\sqrt{5}} + \cdots.$

3. $\dfrac{x}{4} - \dfrac{x^3}{4^2} + \dfrac{x^5}{4^3} - \dfrac{x^7}{4^4} + \cdots.$

4. $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots.$

5. $1 + \dfrac{x}{5} + \dfrac{x^2}{2\cdot 5^2} + \dfrac{x^3}{3\cdot 5^3} + \cdots.$

6. $1 + \dfrac{x}{1^2\cdot 2} + \dfrac{x^2}{3^2\cdot 2^2} + \dfrac{x^3}{5^2\cdot 2^3} + \dfrac{x^4}{7^2\cdot 2^4} + \cdots.$

7. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots.$

8. $1 + \dfrac{x^2}{4} + \dfrac{x^4}{7} + \dfrac{x^6}{10} + \cdots.$

9. $1 + \dfrac{2x}{3} + \dfrac{3x^2}{9} + \dfrac{4x^3}{27} + \cdots.$

10. $1 + x + 2!\,x^2 + 3!\,x^3 + \cdots.$

11. $(x-3) - 2(x-3)^2 + 3(x-3)^3 - 4(x-3)^4 + \cdots.$

12. $(x-1)+\dfrac{(x-1)^2}{2\cdot 2!}+\dfrac{(x-1)^3}{3\cdot 3!}+\cdots.$

13. $(x+1)-\frac{1}{2}(x+1)^2+\frac{1}{3}(x+1)^3-\frac{1}{4}(x+1)^4+\cdots.$

14. $(x-4)-\frac{1}{4}(x-4)^2+\frac{1}{9}(x-4)^3-\frac{1}{16}(x-4)^4+\cdots.$

15. $\dfrac{x-2}{1\cdot 2}-\dfrac{(x-2)^2}{3\cdot 4}+\dfrac{(x-2)^3}{5\cdot 6}-\dfrac{(x-2)^4}{7\cdot 8}+\cdots.$

16. $1+(x-2)+\dfrac{(x-2)^2}{2^2}+\dfrac{(x-2)^3}{3^2}+\dfrac{(x-2)^4}{4^2}+\cdots.$

CHAPTER XXVII

EXPANSION OF FUNCTIONS

167. Introduction.—By the simple process of long division and by the use of the binomial theorem, we have already obtained relations such as,

$$(1) \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

$$(2) \quad (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2^2\,2!}x^2 + \frac{3}{2^3\,3!}x^3 - \frac{3\cdot 5}{2^4\,4!}x^4 + \cdots .$$

Both series are convergent in the interval $-1 < x < 1$. Furthermore, it can be shown that for any value of x in this interval, *the sum of either series is the value of the function from which the series was obtained.* The series may, therefore, be said to *represent* the function for values of x in this interval; it may be used to compute the value of the function for any such value of x. Thus, putting $x = \frac{1}{9}$ in (2) we have

$$\frac{\sqrt{10}}{3} = 1 + \frac{1}{2(9)} - \frac{1}{2!\,2^2 9^2} + \frac{3}{3!\,2^3 9^3} - \frac{3\cdot 5}{4!\,2^4 9^4} + \cdots$$

from which $\sqrt{10}$ can be computed with any desired degree of accuracy.

We now ask whether other functions such as $\sin x$, e^x, and $\log(1+x)$ can be similarly represented by power series; specifically, "Is it possible, by getting the proper values for $C_0, C_1, C_2, \cdots$, to write

$$\sin x = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

in the same sense that (1) and (2) above are written?"

If it is possible, how does one determine the values for the C's?

Assuming that such an "expansion" of a given function in a power series is possible, we shall show in the next article how the values of the C's are found. For this purpose we shall need the following theorem which will be stated without proof:

Theorem: *If the power series* $C_0 + C_1x + C_2x^2 + \cdots$ *converges in a certain interval and represents a function* $f(x)$, *then the series obtained by differentiating this power series term by term will converge in the same interval (with the possible exception of the end points), and will represent the derivative* $f'(x)$ *of this function.*

168. Maclaurin's series.—Assume that for all values of x in an interval of the form $-r < x < +r$, the power series $C_0 + C_1x + C_2x^2 + \cdots$ converges and represents the function $f(x)$; *i.e.*, assume that

$$(1) \quad f(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \cdots .$$

We wish then to determine how the C's are related to the function $f(x)$. Putting $x = 0$ in (1) we have immediately

$$f(0) = C_0;$$

i.e., the first constant C_0 must be equal to the value of $f(x)$ when $x = 0$; next, differentiating (1) and then putting $x = 0$, we have

$$f'(x) = C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + \cdots ,$$
$$f'(0) = C_1.$$

Differentiating again and putting $x = 0$, we get

$$f''(x) = 2C_2 + 2 \cdot 3C_3x + 3 \cdot 4C_4x^2 + \cdots ,$$
$$f''(0) = 2C_2, \qquad \text{or} \qquad C_2 = \frac{f''(0)}{2}.$$

Repeating the process we find next,

$$f'''(x) = 2 \cdot 3C_3 + 2 \cdot 3 \cdot 4C_4x + \cdots ,$$
$$f'''(0) = 2 \cdot 3C_3, \qquad \text{or} \qquad C_3 = \frac{f'''(0)}{3!}.$$

Proceeding in this way we see that in general

$$C_n = \frac{f^{(n)}(0)}{n!}.$$

Substituting these values for the C's in (1), we have

$$(2)\quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots.$$

This is called *Maclaurin's series.* It can obviously be formally written down for any function which is defined at $x = 0$ and has derivatives of all orders at this point. It must be emphasized however that we have not *proved* that the series represents the function. We have merely shown that if $f(x)$ can be represented by a series of the form (1), then the C's must have the values given in (2). Cases can be found in which the sum of the series is not equal to the value of the function, but none will be encountered in our elementary work.

Example

Expand the function $\sin x$ in a Maclaurin series.

Solution

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0\\
f'(x) &= \cos x & f'(0) &= 1\\
f''(x) &= -\sin x & f''(0) &= 0\\
f'''(x) &= -\cos x & f'''(0) &= -1\\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1\\
&\cdots\cdots & &\cdots\cdots
\end{aligned}$$

Substituting in (2) the values found above in the right-hand column, we have

$$\sin x = 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

or

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

The series can easily be shown to converge for all values of x. It may then be used to compute $\sin x$ for any value of x—and since it is an alternating series, the error made in stopping at any term is less than the first neglected term.

PROBLEMS

The following five expansions are particularly important. Verify both the series and the interval of convergence.

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all values of x.

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all values of x.

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all values of x.

4. $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x \leqq +1.$

5. $\text{arc} \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad -1 \leqq x \leqq +1.$

Expand each of the following functions in a Maclaurin series giving four terms and the nth term; find the interval of convergence.

6. a^x.
7. $\log (1 - x)$.
8. $\frac{1}{1 + x}$.
9. $\sin \left(x - \frac{\pi}{4}\right)$.
10. $\sqrt{1 + x}$.
11. $(1 + x)^m$.

Expand each of the following, giving three terms. Note that it would be difficult to formulate the nth term.

12. $\tan x$.
13. $\sec x$.
14. $\log (\sec x + \tan x)$.
15. $\log \cos x$.

16. From the series given above for e^x write down that for e^{-x}; obtain also that for $\frac{1}{2}(e^x + e^{-x})$ assuming that it is permissible to add two series, term by term.

17. Approximate the number e from the above series for e^x using 10 terms. Obtain an upper limit for the error. HINT: The error is

$$\frac{1}{10!} + \frac{1}{11!} + \frac{1}{12!} + \cdots = \frac{1}{10!}\left(1 + \frac{1}{11} + \frac{1}{11 \cdot 12} + \cdots\right)$$
$$< \frac{1}{10!}\left(1 + \frac{1}{11} + \frac{1}{11^2} + \cdots\right)$$

18. Explain why $\log x$ cannot be expanded in a Maclaurin series. What about $x^{\frac{1}{3}}$?

19. Show that the series for $\cos x$ is obtained if one differentiates that for $\sin x$ term by term.

20. In studying the motion of a pendulum and in other investigations, one often replaces sin θ by θ if θ is small. Justify this approximation. For what size angles could one be certain that the error involved is less than 0.0005?

21. Discuss the accuracy with which $\cos x$ may be replaced by $1 - \frac{1}{2}x^2$ for x between -0.1 and $+0.1$.

22. Expand the function $\frac{1}{1-x}$ in powers of x using (*a*) long division, (*b*) the binomial theorem, (*c*) Maclaurin's formula. Does the function appear to have only *one* expansion in powers of x?

169. Taylor's series.—A function $f(x)$ cannot be expanded in a Maclaurin series unless it is defined and has derivatives of all orders *at the point where* $x = 0$; thus, such functions as $\log x$ and $\sqrt{x}$, cannot be so expanded.

It is fairly obvious also that the Maclaurin series, when it can be obtained, is useful in computing the value of $f(x)$ *primarily for values of* x *near* 0; for it is for such values of x that the terms involving x^3, x^4, x^5, etc., decrease rapidly.

We shall now derive a formula for the expansion of a function in powers of $(x - a)$ where a is a constant which may be selected arbitrarily; such a series is useful for computing the value of $f(x)$ for values of x near a. The procedure is the same as that used in deriving Maclaurin's series.

Assume that in some interval of the form $a - r < x < a + r$ the series $C_0 + C_1(x - a) + C_2(x - a)^2 + \cdots$ converges and represents the function $f(x)$; we may then write, for all values of x in this interval which includes the point $x = a$,

$$(3)\ f(x) = C_0 + C_1(x - a) + C_2(x - a)^2 + C_3(x - a)^3 + C_4(x - a)^4 + \cdots.$$

Differentiating successively, we obtain the equations:

$$f'(x) = C_1 + 2C_2(x - a) + 3C_3(x - a)^2 + 4C_4(x - a)^3 + \cdots;$$

$$f''(x) = 2!C_2 + 3!C_3(x - a) + 4 \cdot 3C_4(x - a)^2 + \cdots;$$

$$f'''(x) = 3!C_3 + 4!C_4(x - a) + \cdots;$$

. . . .

Substituting $x = a$ in these equations and solving for the C's, we obtain

$$C_0 = f(a), \qquad C_1 = f'(a), \qquad C_2 = \frac{f''(a)}{2!}, \qquad C_3 = \frac{f'''(a)}{3!},$$

and we can easily see that

$$C_n = \frac{f^{(n)}(a)}{n!}.$$

Substituting these values for the C's in (3) we have

$$(4) \quad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

This is called Taylor's series. It is the expansion of $f(x)$ in powers of $(x - a)$ and is, in general, valid in an interval which is bisected by the point $x = a$. It converges most rapidly of course for values of x *near* a because for such values the terms involving $(x - a)^2$, $(x - a)^3$ etc., decrease rapidly. The Maclaurin series is obviously a special case of this more general expansion; it is obtained by taking $a = 0$.

Example

Expand the function $\log x$ in a Taylor's series taking $a = 1$.

Solution

$$\begin{aligned}
f(x) &= \log x & f(1) &= 0 \\
f'(x) &= \frac{1}{x} & f'(1) &= 1 \\
f''(x) &= -\frac{1}{x^2} & f''(1) &= -1 \\
f'''(x) &= +\frac{2}{x^3} & f'''(1) &= 2! \\
f^{(4)}(x) &= -\frac{3 \cdot 2}{x^4} & f^{(4)}(1) &= -3!
\end{aligned}$$

. . . .

Substituting in (4) the values found above in the right-hand column, we have

$$\log x = 0 + 1(x - 1) + \frac{-1}{2!}(x - 1)^2 + \frac{2}{3!}(x - 1)^3 + \frac{-3!}{4!}(x - 1)^4 + \cdots$$

$$= \frac{x - 1}{1} - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots.$$

The interval of convergence is $0 < x \leqq 2$.

170. Operations with power series.—Operations which one can always perform in dealing with a *finite* number of terms are not necessarily permissible when *infinite* series are involved. Thus, we have already seen that changing the order of the terms in a conditionally convergent series may change its sum or cause it to become divergent.

The following remarks concerning operations with infinite series in general and power series in particular will be found useful. Proofs are omitted.

Addition.—If two convergent series of constant terms having sums U and V, respectively, are added term by term, the resulting series is convergent with sum $U + V$. Consequently, if two power series representing respectively $f(x)$ and $g(x)$ be so added, the resulting series will converge and represent $f(x) + g(x)$ for all values of x for which *both* series are convergent.

Multiplication.—If two convergent series of constant terms having sums U and V, respectively, are multiplied together by the usual rule for multiplying polynomials, the resulting series is convergent with sum $U \cdot V$ provided at least one of the two series is *absolutely* convergent. If both are only conditionally convergent the product series may diverge; if it does converge however its sum will be $U \cdot V$. Any power series is absolutely convergent for all values of x in the *interior* of its interval of convergence. Consequently, if two power series representing $f(x)$ and $g(x)$ respectively be so multiplied, the resulting series will certainly converge and represent $f(x) \cdot g(x)$ for all values of x in the interior of *both* intervals of convergence.

Differentiation and Integration.—For series of variable terms in general, the questions of differentiation and integration involve the conception of "uniform" convergence which cannot be treated here. For the special case of *power series*, however, the necessary theorem concerning differentiation was stated in Art. 167 and was used in deriving both Maclaurin's and Taylor's series. The corresponding statement concerning term by term integration is as follows:

If the power series representing $f(x)$ be integrated term by term between the limits a and b, the resulting series will converge and represent $\int_a^b f(x)dx$ provided both limits are in the **interior** *of the interval of convergence.*

A definite integral may sometimes be evaluated when the indefinite integral cannot be found, by expanding the integrand in a power series and integrating term by term.

Example

Evaluate $\int_0^{\frac{1}{2}} e^{-x^2}dx$.

Solution

We cannot find an indefinite integral; but

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots *.$$

The series converges for all values of x, hence the limits 0 and $\frac{1}{2}$ are inside the convergence interval; we have, then,

$$\begin{aligned}\int_0^{\frac{1}{2}} e^{-x^2}dx &= \int_0^{\frac{1}{2}}\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right)dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\Big]_0^{\frac{1}{2}} \\ &= 0.4613.\end{aligned}$$

Term by term differentiation or integration may also provide an easy method of obtaining the power series for many functions.

* This series is easily obtained by writing down the well-known series for e^x and then replacing x by $-x^2$.

Example

Expand arc tan x in a Maclaurin series.

Solution

The series for $\frac{1}{1+x^2}$, obtained by long division, is

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots, \qquad |x| < 1.$$

Integrating both sides between limits $x = 0$ and $x = x$, we have

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x (1 - x^2 + x^4 - x^6 + \cdots)dx,$$

or,

$$\text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad |x| < 1.$$

PROBLEMS

1. Expand sin x in a Taylor series taking $a = \pi/6$. Is this series better for computing sin 35° than the Maclaurin series?

2. Expand cos x in powers of $x - \frac{\pi}{4}$ to four terms.

3. Expand $\sqrt{x}$ in powers of $x - 1$, giving four terms; find the interval of convergence. Compute $\sqrt{1.5}$ to three decimal places.

4. Expand e^x in powers of $x - 1$ to four terms.

5. Expand log x in powers of $x - 2$ to four terms.

6. Differentiate the Maclaurin series for e^x. Explain the result.

7. Derive the Maclaurin series for $1/(1 - x)^2$ from that for log $(1 - x)$ by differentiating.

8. By long division obtain the Maclaurin series for $\frac{1}{1-x^2}$; from this obtain the series for $\log\frac{1+x}{1-x}$ by integration. Compute log 2 by putting $x = \frac{1}{3}$. $\left(\int \frac{dv}{a^2 - v^2} = \frac{1}{2a}\log\frac{a+v}{a-v} + C\right)$.

9. Expand $1/\sqrt{1 - x^2}$ by the binomial theorem. By integration obtain the series for arc sin x. From this approximate the number π.

10. Approximate the area under the curve $y = e^{-\frac{1}{2}x^2}$ from $x = 0$ to $x = 1$.

11. Obtain the first four terms of the Maclaurin series for $e^x \sin x$ by multiplication. What is the interval of convergence?

12. Obtain the Maclaurin series for log cos x from that for tan x by integration.

171. The remainder in Taylor's series.—In deriving Taylor's series we assumed that it was possible to represent a function $f(x)$ by a power series which converges in some interval about the point $x = a$. Let us now discard this assumption and assume only that $f(x)$ and its first $(n - 1)$ derivatives are defined at $x = a$; with no further assumptions we may, of course, write

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n - 1)!}(x - a)^{n-1} + R_n$$

where R_n denotes the remainder after n terms. This equation merely states that the value of $f(x)$, for *any* value of x, is equal to the sum of the n terms indicated plus *some positive or negative quantity* R_n, which is defined by the equation itself to be whatever is necessary in order that the two sides may balance.

If, for a certain value of x, it happens that

$$\lim_{n \to \infty} R_n = 0,$$

then Taylor's series not only converges but also *represents* $f(x)$ for this value of x. We see, therefore, a possibility of justifying the assumption which was made in deriving Taylor's series.

Our next problem is to obtain a general expression for R_n—not only for the purpose of studying the conditions under which $R_n \to 0$, but also for the purpose of estimating the error involved in approximating the value of $f(x)$ by using the first n terms of Taylor's series. Before deriving this general expression, we must introduce two preliminary theorems.

172. Rolle's Theorem and the Theorem of Mean Value. The first preliminary theorem is

Rolle's Theorem: *If a single-valued function $f(x)$ and its derivative $f'(x)$ are continuous in the interval $a \leqq x \leqq b$, and*

if $f(a) = f(b) = 0$, *then there is at least one value of* $x(=x_1)$ **between a and b** *such that* $f'(x_1) = 0$.

The geometrical meaning of the theorem is shown in Fig. 169. If $f(a) = f(b) = 0$, the graph of $f(x)$ crosses the x-axis at $x = a$ and $x = b$; the theorem then states that at

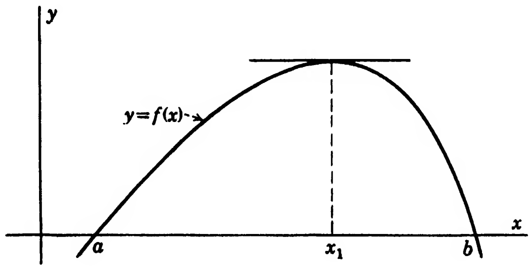

FIG. 169.

some intermediate point x_1 where $a < x_1 < b$, the tangent line must be horizontal. The theorem can be proved analytically, but we shall accept it as geometrically obvious.

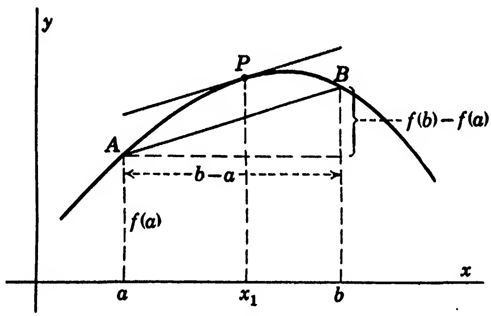

FIG. 170.

The second theorem, of which Rolle's Theorem is a special case, is the

Theorem of Mean Value: *If a single-valued function* $f(x)$ *and its derivative* $f'(x)$ *are continuous in the interval* $a \leqq x \leqq b$, *there exists at least one point* x_1 **between a and b** *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(x_1).$$

The geometrical meaning of this theorem is shown in Fig. 170; the quotient $\frac{f(b) - f(a)}{b - a}$ is the slope of the chord AB; the theorem states that there is a point P, whose abscissa x_1 is *between a and b*, at which the tangent line is parallel to this chord. This is clearly a generalization of Rolle's Theorem and may also be accepted as geometrically obvious. Written in the equivalent form

$$f(b) = f(a) + (b - a)f'(x_1), \qquad a < x_1 < b,$$

the theorem takes on a new meaning. It gives us an expression for the remainder *after one term* in Taylor's series. For, suppose that we had $f(x)$ expanded in such a series about the point $x = a$ and were using the series to compute the value of $f(x)$ at a neighboring point $x = b$ inside the interval of convergence. We should have

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!}f''(a) + \cdots.$$

If we stop after one term we may write

$$f(b) = f(a) + R_1$$

where R_1 is the sum of the remaining series. But, from the theorem of mean value, $f(b)$ is given *exactly* by the expression

$$f(b) = f(a) + (b - a)f'(x_1), \qquad a < x_1 < b.$$

Hence, we have

$$R_1 = (b - a)f'(x_1), \qquad a < x_1 < b.$$

173. Taylor's Theorem.—We wish now to obtain a general expression for the remainder after n terms in Taylor's series. In order to make the procedure clear we shall first give an analytical proof, based on Rolle's Theorem, of the expression just discussed for the remainder after *one* term. Then we shall derive the corresponding expression for the remainder after two terms. Finally we shall state the general theorem.

Remainder after one term.—Assuming that $f(x)$ and its derivative $f'(x)$ are continuous in the interval $a \leqq x \leqq b$, write

$$\text{(I)} \qquad f(b) = f(a) + (b - a)Q$$

where Q is to be determined. In order to find Q form arbitrarily a function $\varphi(x)$ by transposing all terms of (I) to the left side and replacing a by x. This gives

$$\varphi(x) = f(b) - f(x) - (b - x)Q.$$

Now obviously $\varphi(b) = 0$; also $\varphi(a) = 0$ by virtue of (I). Hence, by Rolle's Theorem, $\varphi'(x)$ must equal zero for *some* value of $x(=x_1)$ between a and b. But, differentiating,

$$\varphi'(x) = -f'(x) + Q.$$

Substituting for x the value x_1 for which $\varphi'(x) = 0$, we have

$$0 = -f'(x_1) + Q$$

or

$$Q = f'(x_1).$$

Putting this value of Q in (I) we have

$$f(b) = f(a) + (b - a)f'(x_1) \text{ where } a < x_1 < b.$$

Remainder after two terms.—Assuming that $f(x)$ and its first two derivatives are continuous in the interval $a \leqq x \leqq b$, write

$$\text{(II)} \qquad f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!}Q$$

where Q is to be determined. In order to find Q, form arbitrarily a function $\varphi(x)$ by transposing all terms of (II) to the left side and replacing a by x. This gives

$$\varphi(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b - x)^2}{2!}Q.$$

Now obviously $\varphi(b) = 0$; also $\varphi(a) = 0$ by virtue of (II). Hence, by Rolle's Theorem, $\varphi'(x)$ must equal zero for *some* value of $x(=x_1)$ between a and b. But, differentiating,

$$\begin{aligned}\varphi'(x) &= -f'(x) - (b-x)f''(x) + f'(x) + (b-x)Q \\ &= (b-x)[-f''(x) + Q].\end{aligned}$$

Substituting for x the value x_1 for which $\varphi'(x) = 0$, and noting that since $a < x_1 < b$ the factor $(b - x_1) \neq 0$ we have

$$Q = f''(x_1).$$

Putting this value of Q in (II) we have

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(x_1), \text{ where}$$
$$a < x_1 < b.$$

Proceeding in this way it is possible to prove the important general theorem known as

Taylor's Theorem: *If a single-valued function $f(x)$ and its first n derivatives are continuous in the interval $a \leqq x \leqq b$, then*

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n,$$

where R_n, called the remainder after n terms, is given by the formula

$$R_n = \frac{(b-a)^n}{n!}f^{(n)}(x_1) \textit{ where } a < x_1 < b.$$

The student should notice that this expression for the remainder is merely the next term in the series with $f^{(n)}(a)$ replaced by $f^{(n)}(x_1)$ where in general all that is known about x_1 is that it is between a and b. Without further information concerning x_1 we cannot calculate R_n exactly. We may, however, find an upper bound for the remainder by taking for x_1 the value between a and b which makes the above expression as large as possible.

Example

Compute $\sqrt[3]{e}$ using five terms of Maclaurin's series and estimate the accuracy of the result.

Solution

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Putting $x = \frac{1}{3}$ and using five terms we may write

$$\begin{aligned} e^{\frac{1}{3}} &= 1 + \tfrac{1}{3} + \frac{(\frac{1}{3})^2}{2!} + \frac{(\frac{1}{3})^3}{3!} + \frac{(\frac{1}{3})^4}{4!} + R_5, \\ &= 1.39557 + R_5. \end{aligned}$$

Since in this case $(b - a) = \frac{1}{3}$, and $f^{(5)}(x) = e^x$, we have

$$R_5 = \frac{(\frac{1}{3})^5}{5!} e^{x_1} \text{ where } 0 < x_1 < \tfrac{1}{3}.$$

The quantity e^{x_1} is certainly not larger than $e^{\frac{1}{3}}$, and since $e < 3$ it is smaller than $3^{\frac{1}{3}}$. Hence,

$$R_5 < \frac{(\frac{1}{3})^5\sqrt[3]{3}}{5!} = \frac{\sqrt[3]{3}}{29{,}160} = 0.00005.$$

The value of $e^{\frac{1}{3}}$, correct to four decimal places, is then 1.3956.

PROBLEMS

1. Sketch the parabola $y = x^2$ and show two points $A(a, a^2)$ and $B(b, b^2)$ on it; prove that the tangent to the curve drawn at the point where $x = \dfrac{a+b}{2}$ is parallel to the chord AB.

2. Find the point P on the curve $y = x^3$ at which the tangent line is parallel to the chord joining the points where $x = 1$ and $x = 4$.

3. Sketch the curve $y = x^3 - 3x^2 - 6x + 8$. Find a point $P(x_1, y_1)$ on the curve with $1 < x_1 < 4$ such that the tangent at P is parallel to the chord joining the points where $x = 1$ and $x = 4$. Is this an illustration of Rolle's theorem or the Theorem of Mean Value or both?

4. Suppose that $f(x)$ is expanded in a Taylor's series about the point $x = a$. Show that using the first two terms in approximating $f(b)$ is equivalent to using the differential of $f(x)$ as an approximation to its increment. Illustrate with a sketch.

5. Prove that the remainder in Taylor's series after three terms is $\dfrac{(b-a)^3}{3!} f'''(x_1)$ where $a < x_1 < b$.

6. Write out the special case of Taylor's theorem for a Maclaurin series.

7. Approximate $\sqrt{e}$ from Maclaurin's series using six terms. Estimate the error both by using Taylor's Theorem and by comparing the remainder series with a geometric series.

8. Using four terms of the Taylor's series for log x about the point $x = 1$, calculate log 1.3. Estimate the error.

9. Expand sin x in powers of $x - \frac{\pi}{6}$; compute sin 36° using three terms; estimate the error.

10. Compare the accuracy with which sin 36° can be computed using three terms of Taylor's series about the point $x = \pi/6$ with that involved in using three terms of Maclaurin's series.

174. Relations between the exponential and trigonometric functions.—We have already derived the following series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots; \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots; \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{3}$$

We have seen that all of these series are convergent for all real values of x. It can also be shown that the remainder R_n approaches zero as n increases indefinitely, no matter what value is assigned to x. Each series therefore represents the corresponding function for all real values of x.

No meaning has so far been attached to the symbol e^x if x is an imaginary or complex number. It is shown in more advanced works however that the above series is convergent for all complex as well as real values of x. We may, therefore, agree arbitrarily to let the exponential function be defined by this series for all real *and complex* values of the variable; *i.e.*, if z is any complex number ($= a + bi$), the value of e^z is *by definition* given by the convergent series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \tag{4}$$

With the exponential function thus defined for complex values of z, we may now derive certain remarkable relations between the trigonometric functions of a real angle (or

number) x and the exponential functions of the corresponding imaginary number ix as follows: Substituting $z = ix$ (where $i = \sqrt{-1}$) in (4) we have

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots.$$

Using the relations $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc., this can be reduced to

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).^*$$

Since the series in the parentheses represent $\cos x$ and $\sin x$ respectively we have the relation

(5) $$e^{ix} = \cos x + i \sin x.$$

Replacing x by $-x$ we have also

(6) $$e^{-ix} = \cos x - i \sin x.$$

Finally, by adding and subtracting these two equations, we find that

(7) $$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

(8) $$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

The relations (5), (6), (7), and (8) are of great importance in many of the applications of mathematics. They enable one to change exponential functions of the imaginary number ix over into trigonometric functions of the real number x and vice versa. The student will have occasion to use them in the next chapter for the purpose of simplifying the solutions of certain differential equations.

With the exponential function e^z defined for complex values of z by series (4), we define the sine and cosine of the complex quantity z in accordance with (7) and (8) as

* The rearrangement of terms is permissible because the series is *absolutely* convergent for all values of x.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

If now we let $z = ix$ where x is real, we have

$$\sin (ix) = i\frac{e^{x} - e^{-x}}{2} \tag{9}$$

$$\cos (ix) = \frac{e^{x} + e^{-x}}{2}. \tag{10}$$

These equations express the trigonometric functions of the imaginary quantity ix in terms of the exponential functions of the real number x.

175. The hyperbolic functions.—The exponential functions

$$\frac{e^{x} - e^{-x}}{2} \quad \text{and} \quad \frac{e^{x} + e^{-x}}{2},$$

which appear in equations (9) and (10) of the preceding section, occur frequently in many branches of applied mathematics—so frequently in fact that it has been deemed useful to give them special names and make special studies of their properties. From the way in which they occur in (9) and (10) it might be suspected that they have some properties which are similar to those of $\sin x$ and $\cos x$. It can in fact be shown that they are related to the equilateral hyperbola in somewhat the same way that the trigonometric functions are related to the circle. The names chosen for them are accordingly the *hyperbolic sine* of x and the *hyperbolic cosine* of x respectively. The abbreviations usually used are $\sinh x$ and $\cosh x$. We have then, *by definition,*

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \tag{1}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}. \tag{2}$$

The *hyperbolic tangent* of x, is defined by the relation $\tanh x = \dfrac{\sinh x}{\cosh x}$ and the hyperbolic cotangent, secant, and

cosecant, are defined respectively as the reciprocals of the hyperbolic tangent, cosine, and sine. The values of sinh x, cosh x, and tanh x, and their logarithms, are given in tabular form in various books of tables. (See Table V in this book.)

The fundamental properties of the hyperbolic functions are deduced directly from their definitions. As already remarked, many of these properties are analogous to those of the trigonometric functions.

Example 1

Show that $\frac{d}{dx}\sinh x = \cosh x$.

Solution

$$\begin{aligned}\sinh x &= \tfrac{1}{2}(e^x - e^{-x})\\ \frac{d}{dx}\sinh x &= \tfrac{1}{2}(e^x + e^{-x})\\ &= \cosh x.\end{aligned}$$

Example 2

Show that $2\sinh x \cosh x = \sinh 2x$.

Solution

$$\begin{aligned}2\sinh x\cosh x &= 2\cdot\tfrac{1}{2}(e^x - e^{-x})\cdot\tfrac{1}{2}(e^x + e^{-x})\\ &= \tfrac{1}{2}(e^{2x} - e^{-2x})\\ &= \sinh 2x.\end{aligned}$$

Example 3

Show that $\cosh^2 x - \sinh^2 x = 1$.

Solution

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}\\ &= 1.\end{aligned}$$

The student should be careful to note that while some of the properties of the hyperbolic functions correspond exactly to those of the circular functions, some of the

relations contain important differences. Thus, for the circular functions, we have $\cos^2 x + \sin^2 x = 1$, while for the hyperbolic functions the corresponding relation is $\cosh^2 x - \sinh^2 x = 1$. Further important relations are brought out in the following set of problems.

PROBLEMS

1. From equations (5) and (6) of Art. 174, derive the relation $e^{\pm\pi i} = -1$; show also that $e^{2k\pi i} = 1$, if k is any integer.

2. Show that $e^{(a\mp bi)x} = e^{ax}(\cos bx \mp i\sin bx)$.

3. Prove the relations

$$\begin{cases}\sin(ix) = i\sinh x\\ \cos(ix) = \cosh x\end{cases}\qquad \begin{cases}\sinh(ix) = i\sin x\\ \cosh(ix) = \cos x.\end{cases}$$

4. Prove the identities

(a) $$\cosh^2 x - \sinh^2 x = 1$$
(b) $$\tanh^2 x + \operatorname{sech}^2 x = 1$$
(c) $$\coth^2 x - \operatorname{csch}^2 x = 1.$$

5. Prove the relations

(a) $$\cosh x + \sinh x = e^x$$
(b) $$\cosh x - \sinh x = e^{-x}.$$

6. Show that

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$
$$\tanh(-x) = -\tanh x.$$

7. Show that

(a) $$\sinh 2x = 2\sinh x\cosh x$$
(b) $$\cosh(2x) = \cosh^2 x + \sinh^2 x$$
(c) $$\tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}.$$

8. Show that

(a) $$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh y$$
(b) $$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y$$
(c) $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x\tanh y}.$$

9. Show that

(a) $$\frac{d}{dx}(\sinh u) = \cosh u\frac{du}{dx}$$

(b) $$\frac{d}{dx}(\cosh u) = \sinh u\frac{du}{dx}$$

(c) $$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u\frac{du}{dx}.$$

Differentiate the following functions using the formulas of Prob. 9:

10. $y = \cosh 2x$.

11. $y = x \sinh x$.

12. $y = e^{-x} \sinh x$.

13. $y = \log \cosh x$.

14. $y = \tanh^2 x$.

15. $y = \sinh^2 2x$.

16. Sketch on the same axes the functions e^x and e^{-x}; from these obtain sketches of the curves $y = \cosh x$ and $y = \sinh x$ by composition of ordinates.

17. Show that the equation $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, previously given for the catenary, is equivalent to $y = a \cosh \frac{x}{a}$. Sketch the curve. This is the curve in which a homogeneous cable hangs when suspended from two points and acted on only by its own weight.

18. Show that for all values of x

(a) $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

(b) $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

CHAPTER XXVIII

DIFFERENTIAL EQUATIONS

176. Definitions.—A differential equation is an equation containing differentials or derivatives. The following are typical examples:

$$\frac{d^2y}{dx^2} - 6x = 0; \tag{1}$$

$$x\frac{dy}{dx} + 3y = y^3; \tag{2}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + 6\left(\frac{dy}{dx}\right)^3 = y. \tag{3}$$

These equations all contain derivatives. When only the first derivative is involved, the relation is frequently written in a form containing differentials instead. Thus, (2) may be written in the form

$$x\,dy = (y^3 - 3y)dx.$$

The **order** of a differential equation is that of the *highest ordered derivative* occurring in it. Thus, (1) and (3) above are of second order while (2) is of first order. The *exponent of the derivative of highest order* gives the **degree** of the differential equation. Thus, (1) and (2) are of first degree while (3) is of second degree.

177. Solutions of differential equations.—A solution of a differential equation is any relation between the variables involved which satisfies the equation. The student has already solved simple differential equations of the form

$$\frac{d^ny}{dx^n} = f(x)$$

by direct integration. He is familiar with the fact that such an equation has a solution which contains n arbitrary

constants. Example 1 in Art. 176 is of this type. Writing it in the form

$$\frac{d^2y}{dx^2} = 6x$$

we have, after integrating twice,

$$y = x^3 + C_1x + C_2.$$

This result is called the *general solution* of the given differential equation. Any solution which can be obtained from it by giving particular values to C_1 and C_2 is called a *particular solution.*

In general, any differential equation of order n has a solution called the *general solution* which contains exactly n essential arbitrary constants. Usually it has no other solution except the *particular solutions* obtainable from this by giving particular values to the constants. There are important exceptional cases when equations of degree higher than the first are considered; such cases will not however be treated here.

There is no general procedure for solving a differential equation. Only a few types can be solved at all, and we shall treat in this chapter only the simplest of these—taking, in general, those which occur often in various elementary applications. When a solution, or what is supposed to be a solution, has been obtained by any method, its correctness can of course be determined by direct substitution.

Example

Show that $y = A \sin x + B \cos x$ is the general solution of the equation $\dfrac{d^2y}{dx^2} + y = 0.$

Solution

By differentiating twice we find that, if

$$y = A \sin x + B \cos x,$$

$$\frac{d^2y}{dx^2} = -A \sin x - B \cos x;$$

hence,

$$\frac{d^2y}{dx^2} + y = 0.$$

The given relation is then a solution of the differential equation, and since it contains two arbitrary constants, it is the general solution.

178. Separation of variables.—Any differential equation of first order and first degree can be written in the form

$$M\,dx + N\,dy = 0$$

where in general M and N are functions of both x and y. It is often possible to transform the equation so as to get all of the terms containing x together with dx on one side, and all of those containing y together with dy on the other. When this can be done, the general solution can be obtained by integrating the two sides separately and adding the arbitrary constant on either side. This case was discussed briefly in Chap. XV in connection with elementary applications of the indefinite integral.

Example

$$(1 + x^2)dy - xy\,dx = 0.$$

Solution

Dividing by $y(1 + x^2)$ and transposing, this becomes

$$\frac{dy}{y} = \frac{x\,dx}{1 + x^2}.$$

Integrating both sides, we have

$$\log y = \tfrac{1}{2}\log(1 + x^2) + \log C,$$

or

$$\log y = \log C\sqrt{1 + x^2},$$

from which

$$y = C\sqrt{1 + x^2}.$$

The arbitrary constant was added in the form "log C" merely for convenience.

PROBLEMS

1. Give the order and degree of each of the following differential equations:

(a) $$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y.$$

(b) $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0.$$

(c) $$\left(\frac{dy}{dx}\right)^3 + y = \left(\frac{d^2y}{dx^2}\right)^2.$$

2. Show that $y = e^x \sin x$ is a solution of the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. Is it the general solution?

3. Using the illustrative example in Art. 177, find the equation of the curve which has at every point, $d^2y/dx^2 = -y$, and which goes through $P(0, 2)$ at 45°. Sketch the curve.

4. Show that $y = Ae^{2x} + Be^{3x}$ is the general solution of the equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

5. Show that $y^2 = 4x$ is a particular solution of the equation $dy/dx = 2/y$. Find the general solution by separating the variables and integrating. Interpret the result geometrically.

6. Show that $x^2 + y^2 = Cx$ is the general solution of the equation $y^2 - x^2 = 2xy\frac{dy}{dx}$. Interpret this geometrically.

Solve the following differential equations by separating the variables:

7. $y\,dx + x\,dy = 0$.

8. $x^2\frac{dy}{dx} + y^2 = 0$.

9. $y^2dx = (x + 1)dy$.

10. $\frac{dy}{dx} + y^2 = 1$.

11. $(1 - x)y\,dx + (1 + y)x\,dy = 0$.

12. $(1 + x^2)y\,dy = (1 + y^2)x\,dx$.

13. $L\frac{di}{dt} + Ri = E$.

14. $\sqrt{1 - y^2}\,dx = x^2y\,dy$.

15. $x\,dy + 2y\,dx = xy\,dy$.

179. Homogeneous equations.—A polynomial in x and y is said to be *homogeneous* if all of its terms are of the same degree in x and y taken together; thus

$x^2 - 3xy + 4y^2$ is homogeneous of degree 2;
$x^3y + x^2y^2 - 2y^4$ is homogeneous of degree 4;
$6x - 3y$ is homogeneous of degree 1.

More generally, any function of x and y is said to be *homogeneous of degree n* if the result of replacing x and y

respectively by kx and ky is the same function multiplied by k^n; thus, the function

$$x^2e^{\frac{y}{x}}$$

is homogeneous of degree 2.

A differential equation of the form

$$M\,dx + N\,dy = 0$$

is said to be homogeneous *if M and N are homogeneous functions of x and y of the same degree.* Such an equation can be transformed into an equation in which the variables are separable by the substitution

$$y = vx,$$

where v is a new variable. Differentiating $y = vx$ gives

$$dy = v\,dx + x\,dv;$$

this quantity must be substituted for dy when vx is substituted for y.

Example

$$(x^2 - y^2)dx + 2xy\,dy = 0.$$

We cannot separate the variables, but M and N are homogeneous functions, both of degree 2. Substituting

$$y = vx \qquad \text{and} \qquad dy = v\,dx + x\,dv$$

we get

$$(1 - v^2)dx + 2v(v\,dx + x\,dv) = 0.$$

Separating the variables and integrating, we have

$$\frac{2v\,dv}{v^2 + 1} = -\frac{dx}{x},$$

$$\log(v^2 + 1) = -\log x + \log C,$$

$$x(v^2 + 1) = C;$$

finally, since $v = y/x$, this reduces to

$$x^2 + y^2 = Cx.$$

The reason for this substitution is apparent when the given equation is written in the form

$$\frac{dy}{dx} = -\frac{M}{N};$$

for if M and N are homogeneous functions of the *same degree*, the right-hand side becomes a function of v alone when vx is substituted for y; *i.e.*, the x's all cancel out. The equation then takes the form

$$v + x\frac{dv}{dx} = \varphi(v)$$

and the variables are obviously separable.

PROBLEMS

Solve the following equations:

1. $x\,dy + (x + y)dx = 0.$
2. $2xy\,dy = (x^2 + 3y^2)dx.$
3. $(x^2 + y^2)dx + xy\,dy = 0.$
4. $\dfrac{dy}{dx} = \dfrac{y^2}{xy - x^2}.$
5. $y^2dx = (x^2 + 2xy)dy.$
6. $(x^3 - y^3)dx + xy^2dy = 0.$
7. $(2x - y)dy = (x - 2y)dx.$
8. $(x - \sqrt{xy})dy = y\,dx.$

9. Find the equation of the curve through (1, 0) which has at every point $\dfrac{dy}{dx} = \dfrac{x + y}{x}$. Sketch the curve.

10. What curve through (1, 1) has at every point $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}$?

11. Show that the equation $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ is made homogeneous by letting $x = x' + h$ and $y = y' + k$, where (h, k) is the point of intersection of the lines represented by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

12. Using the method suggested in Prob. 11, solve the equation $\dfrac{dy}{dx} = \dfrac{x - y - 4}{x + y - 2}$.

180. Exact differential equations.—It may happen that the left-hand side of the equation

$$M\,dx + N\,dy = 0$$

is exactly the total differential of some function $f(x, y)$. In this case the solution is of course

$$f(x, y) = C.$$

Thus, the left-hand side of the equation

$$y^2dx + 2xy\,dy = 0$$

is exactly the differential of the function y^2x; the equation may be written in the form

$$d(y^2x) = 0,$$

and the solution is

$$y^2x = C.$$

An equation of this type is called an *exact differential equation*. Since ordinarily one cannot determine by inspection whether or not a given equation is exact, some test for exactness is necessary. Such a test can easily be obtained as follows: The total differential of any function $f(x, y)$ is

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

If the expression

$$M\,dx + N\,dy$$

is the differential of $f(x, y)$, then obviously

$$\frac{\partial f}{\partial x} = M \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N.$$

Differentiating the first of these with respect to y and the second with respect to x, we have

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial M}{\partial y}, \qquad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x}.$$

But these two second partial derivatives are identically equal for all values of x and y for which they are continuous. The condition which must be satisfied, then, in order that

$$M\,dx + N\,dy = 0$$

be an exact differential equation, is

$$\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x}.$$

Example

Test the equation $(3x^2y - y)dx + (x^3 - x + 2y)dy = 0$ for exactness. Solve the equation if it is exact.

Solution

$$M = 3x^2y - y, \qquad N = x^3 - x + 2y,$$
$$\frac{\partial M}{\partial y} = 3x^2 - 1. \qquad \frac{\partial N}{\partial x} = 3x^2 - 1.$$

Since $\partial M/\partial y = \partial N/\partial x$ the equation is exact; *i.e.*, there is a function $f(x, y)$ of which the left-hand side of the given equation is exactly the total differential. To find this function f we note that

$$\frac{\partial f}{\partial x} = M = 3x^2y - y.$$

Integrating with respect to x, holding y constant, we have

(I) $$f(x, y) = x^3y - yx + \varphi(y)$$

where $\varphi(y)$ consists of terms which are free from x. Similarly, since

$$\frac{\partial f}{\partial y} = N = x^3 - x + 2y,$$

we have upon integrating with respect to y, holding x constant,

(II) $$f(x, y) = x^3y - xy + y^2 + \Psi(x)$$

where $\Psi(x)$ consists of terms which are free from y.

Now, comparing (I) and (II), we see finally that

$$f(x, y) = x^3y - xy + y^2 + C.$$

The solution of the given equation is then

$$x^3y - xy + y^2 + C = 0.$$

From the way in which this solution was obtained, it is evident that *if a given equation is exact* its solution can usually be obtained as follows:

1. *Integrate $M\,dx$ with respect to x, holding y constant.*

2. *Integrate with respect to y only those terms in $N\,dy$ which are free from x.*

3. *Add the terms obtained in step 2 to the result of step 1 and equate the sum to a constant.*

Since in exceptional cases this procedure may give an incorrect result, the solution should always be checked by differentiation.

181. Integrating factors.—It can be shown that if the equation

$$M\,dx + N\,dy = 0$$

is not exact, it can always be made exact by multiplying through by some proper function of x and y. Such a function is called an *integrating factor*. There is no general method of finding an integrating factor, but in many simple cases one can be found by inspection. Consider, for example, the equation

$$y\,dx + 2x\,dy = 0,$$

which is not exact since $\partial M/\partial y \neq \partial N/\partial x$. Multiplying by y we get the exact equation

$$y^2dx + 2xy\,dy = 0,$$

in which the left-hand side is exactly the differential of xy^2; y is then an integrating factor.

PROBLEMS

Show that each of the following equations is exact and find its general solution:

1. $(2x + y)dx + (x - 1)dy = 0.$

2. $(3x^2 + 2y)dx + (2y + 2x + 1)dy = 0.$

3. $(y \cos x + 1)dx + \sin x\,dy = 0.$

4. $\left(2xy + \dfrac{1}{x}\right)dx + x^2dy = 0.$

5. $3x^2y\,dx + \left(x^3 + \dfrac{1}{y}\right)dy = 0.$

6. $\dfrac{dy}{dx} = \dfrac{3x^2y - y}{x - x^3}.$

7. $\dfrac{dy}{dx} = \dfrac{y + x - 1}{2y - x + 2}.$

8. $\dfrac{dy}{dx} = \dfrac{2x^2 + y^2 - 2xy}{x^2 - 2xy}.$

9. Show that the equation $(x^2 + y^2)dx + 2xy\,dy = 0$ is both exact and homogeneous. Solve it by both methods. Which method is easier?

10. Show that the equation $(x^2 - y^2)dx + 2xy\,dy = 0$ is not exact and solve it by another method.

11. Solve the equation $\dfrac{dy}{dx} = \dfrac{y + x}{2y - x}$ by two methods.

12. Show that the equation $\frac{dy}{dx} = \frac{1}{2xy + 8y}$ can be solved either as an exact equation or by separating the variables. Are the methods equivalent?

13. Show that an equation in which the variables are separated is necessarily exact, and that the method used in solving such equations is the same as that given for exact equations.

Show that the following equations are not exact, and try to solve them using an integrating factor.

14. $y\,dx - x\,dy = 0.$ **15.** $x\,dy - y\,dx = x\,dx.$

16. $y\,dx - x\,dy + xy^2dx = 0.$ **17.** $dy - \frac{2y}{x}dx = x^3dx.$

18. Solve the equation $y\,dx + (2x + 2)dy = 0$, both by separating the variables and by using an integrating factor.

182. Linear equations of first order.—A differential equation is said to be *linear* if it is of first degree in the dependent variable y and its derivatives. Such an equation of *first order* can always be written in the form

$$\text{(I)} \qquad \frac{dy}{dx} + Py = Q$$

where P and Q are functions of x alone. It can be solved by finding an integrating factor $R(x)$ such that when both sides are multiplied by R, the left side becomes exactly the derivative of Ry.

To find such a factor, let us multiply both sides of (I) by R, where it is understood that R is a function of x yet to be determined; we get

$$\text{(II)} \qquad R\frac{dy}{dx} + RPy = RQ.$$

Now the derivative with respect to x of Ry is

$$R\frac{dy}{dx} + y\frac{dR}{dx}.$$

The left-hand side of (II) is then the derivative of Ry provided R is a function such that

$$y\frac{dR}{dx} = RPy;$$

separating the variables and solving for R, we have

$$\frac{dR}{R} = P\,dx,$$

$$\log R = \int P\,dx,$$

$$R = e^{\int P\,dx}. \tag{III}$$

When both sides of (I) are multiplied by the function R given by (III), the left-hand side becomes exactly the derivative of Ry. The right-hand side is still a function of x alone. Integrating both sides, we obtain the solution

$$Ry = \int RQ\,dx.$$

Example

$$\frac{dy}{dx} + \frac{2y}{x} = 6x^3.$$

Solution

In this case $P = 2/x$;

$$\int P\,dx = \int \frac{2\,dx}{x} = 2\log x = \log x^2.$$

$$R = e^{\int P\,dx} = e^{\log x^2} = x^2.$$

Multiplying both sides of the given equation by this factor and integrating, we have

$$\begin{aligned} x^2y &= \int 6x^5dx \\ &= x^6 + C. \end{aligned}$$

183. Bernoulli's equation.—The equation

$$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are functions of x alone is known as Bernoulli's equation, after (James) Bernoulli (1654–1705). It can be transformed into a linear equation by letting

$$y = z^{\frac{1}{1-n}}$$

where z is a new variable,

Example

$$\frac{dy}{dx} - \frac{y}{x} = x^2y^2.$$

Solution

Letting $y = z^{\frac{1}{1-2}} = z^{-1}$, the equation becomes

$$\frac{dz}{dx} + \frac{1}{x} \cdot z = -x^2$$

which is linear. An integrating factor is

$$e^{\int P\,dx} = e^{\int \frac{dx}{x}} = e^{\log x} = x.$$

Multiplying by this factor and integrating, we get

$$xz = \int -x^3 dx = -\frac{x^4}{4} + C.$$

Finally, since $z = y^{-1}$, the solution is

$$\frac{x}{y} + \frac{x^4}{4} = C.$$

PROBLEMS

Solve the following equations:

1. $\frac{dy}{dx} + 2y = x.$

2. $\frac{dy}{dx} + y = e^{-x}.$

3. $\frac{dy}{dx} = x + y.$

4. $x\frac{dy}{dx} + y = x^2.$

5. $\frac{dy}{dx} - \frac{2y}{x} = \frac{x+1}{x}.$

6. $\frac{dy}{dx} + \frac{y}{x} = \frac{\sin x}{x}.$

7. $\frac{dy}{dx} + \frac{y}{x} = \sin x.$

8. $\frac{dy}{dx} + y = \sin x.$

9. $\frac{dy}{dx} - \frac{xy}{x^2+1} = x.$

10. $x^2\frac{dy}{dx} - 2xy = 1.$

11. $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}.$

12. $\frac{dy}{dx} = \frac{e^x - 2xy}{x^2}.$

13. $\frac{dy}{dx} = \frac{3 - 2xy}{x^2}.$

14. $x\frac{dy}{dx} + y = e^x - xy.$

15. Show that Bernoulli's equation

$$\frac{dy}{dx} + Py = Qy^n$$

is transformed by letting $y = z^{\frac{1}{1-n}}$ $(n \neq 1)$ into the linear equation

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q.$$

Discuss the cases in which $n = 0$ and $n = 1$.

Solve the following equations:

16. $\dfrac{dy}{dx} + \dfrac{2y}{x} = xy^2$.

17. $3\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{2}{xy^2}$.

18. $\dfrac{dy}{dx} - \dfrac{y}{2} = y^3e^{-x}$.

19. What curve through the origin has its slope at every point equal to the sum of the coordinates of the point?

20. The slope at any point of a curve which goes through (0, 1) is equal to twice the product of the coordinates plus the abscissa. What is its equation?

184. Linear equations of higher order.—In the preceding articles we have given methods for solving certain commonly occurring types of differential equations of *first* order. We shall now consider the most important type of equation of higher order, namely that which is of first degree in the dependent variable y and its derivatives. This so-called linear equation of order n may be written in the form

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = f(x)$$

where the a's may in general be functions of x. We shall consider the equation, however, only for the special case in which the a's are all *constants*.

In studying this equation in the next two articles we shall confine our attention to the case of second order. The student will readily see, however, that the method applies to the general case. For reasons which will soon become obvious, we shall consider first the case in which the right-hand side of the equation is *zero*, and reserve for the following article the more general case in which it is a function of x.

185. Right-hand side zero.—Consider, as an example, the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0.$$

Let us assume that for a proper value of m, the function

$$y = e^{mx}$$

might be a solution. Substituting this "trial solution" into the given equation and solving for m, we have

$$m^2e^{mx} - 2me^{mx} - 3e^{mx} = 0,$$
$$m^2 - 2m - 3 = 0,$$
$$m = -1, \text{ or } 3.$$

The function $y = e^{mx}$ then satisfies the given differential equation if $m = -1$ or $+3$; *i.e.*, $y = e^{-x}$ and $y = e^{3x}$ are two particular solutions. It may easily be shown that $y = Ae^{-x}$ and $y = Be^{3x}$, where A and B are arbitrary constants, are solutions, and finally that

$$y = Ae^{-x} + Be^{3x}$$

is a solution. Since this last solution contains the required number of arbitrary constants, it is the general solution.

This procedure may be used in solving any equation of the form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

where p and q are constants. The assumption that $y = e^{mx}$ is a solution leads to the *auxiliary equation*

$$m^2 + pm + q = 0.$$

If the two roots r_1 and r_2 of this equation are distinct, the general solution of the differential equation is

$$y = Ae^{r_1x} + Be^{r_2x}.$$

The case of equal roots.—If the roots r_1 and r_2 of the auxiliary equation are *equal*, the function

$$y = Ae^{r_1x} + Be^{r_2x} \text{ reduces to } y = Ce^{r_1x}$$

where $C = A + B$. Since it contains only one essential arbitrary constant, it cannot be the general solution. It can be shown that in this case the general solution is

$$y = Ae^{r_1x} + Bxe^{r_1x}.$$

The proof is left as an exercise in the next set.

The case of complex roots.—If the roots of the auxiliary equation are the conjugate complex numbers $a \pm bi$, the general solution is

$$\begin{aligned} y &= Ae^{(a+bi)x} + Be^{(a-bi)x} \\ &= e^{ax}[Ae^{ibx} + Be^{-ibx}]. \end{aligned}$$

This result may be put into a more convenient form by substituting for e^{ibx} and e^{-ibx} the equivalent trigonometric expressions, namely,

$$\begin{aligned} e^{ibx} &= \cos bx + i \sin bx \\ e^{-ibx} &= \cos bx - i \sin bx. \end{aligned}$$

This substitution gives, as an alternate form for the general solution,

$$y = e^{ax}[(A + B) \cos bx + i(A - B) \sin bx]$$

or,

$$y = e^{ax}(C \cos bx + D \sin bx),$$

where C and D are arbitrary constants.

Example

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0.$$

Solution

Assuming a solution of the form $y = e^{mx}$ we obtain the auxiliary equation

$$m^2 - 4m + 13 = 0.$$

The roots of this are $m = 2 \pm 3i$. The general solution is then

$$y = e^{2x}(A \cos 3x + B \sin 3x).$$

PROBLEMS

Solve the following equations:

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0.$

2. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 12y.$

3. $2\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} = 3y.$

4. $\dfrac{d^2y}{dx^2} = 4y.$

5. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0.$

6. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} = 6y.$

7. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} = 4\left(\dfrac{dy}{dx} - y\right).$

8. $4\dfrac{d^3y}{dx^3} - 8\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} - 2y.$

9. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0.$

10. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0.$

11. $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0.$

12. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 0.$

13. $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 26y = 0.$

14. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0.$

15. $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4y.$

16. $\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} = 4y.$

17. $\dfrac{d^4y}{dx^4} = 16y.$

18. At every point of a certain curve the sum of the first and second derivatives is equal to twice the ordinate. The curve goes through the origin with slope 3. What is its equation?

19. What curve passing through (0, 2) and having a horizontal tangent line at that point has at every point $d^2y/dx^2 = y$?

20. What curve passing through the origin at 45° has at every point $\dfrac{d^2y}{dx^2} + y = 2\dfrac{dy}{dx}$? Sketch the curve.

21. What curve passing through (0, 1) at 45° has at every point

$$\frac{d^2y}{dx^2} + 5y = 2\frac{dy}{dx}?$$

22. Find a particular solution of the equation $\dfrac{d^2Q}{dt^2} + 4\dfrac{dQ}{dt} + 5Q = 0,$ satisfying the conditions that $Q = 5$ and $\dfrac{dQ}{dt} = 0$ when $t = 0$.

23. A point starts from rest at (2,0) and moves along the x-axis, its acceleration being always toward the origin and equal to four times the distance from the origin. Find its position and velocity at any time.

24. A chain 4 ft. long is stretched out on a smooth table with 1 ft. of its length hanging over the edge, and then released. Using the relation $F = \frac{w}{g}a$ show that, when any length y (< 4 ft.) is hanging over, the acceleration is $\frac{d^2y}{dt^2} = \frac{g}{4}y$. Find the amount over the edge at the end of t sec.

25. The angular acceleration of a simple pendulum of length L ft. is $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$ where θ is the angular displacement from the vertical. If θ is small, $\sin\theta$ may be replaced by θ with negligible error. With this assumption, and assuming that $\theta = \theta_1$ and $d\theta/dt = 0$ when $t = 0$, express θ in terms of t. Find the time of one complete oscillation.

26. The equation $\frac{d^2y}{dx^2} - 2r\frac{dy}{dx} + r^2y = 0$ has both roots of its auxiliary equation equal to r. Show by direct substitution that $y = Ae^{rx} + Bxe^{rx}$ is the general solution.

186. Right-hand side a function of x.—As an example, consider the equation

$$\text{(I)} \qquad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 6x - 5,$$

which is the same as the first example in Art. 185 with the exception that the right side is now $6x - 5$ instead of 0.

Let us first try to find a particular solution. It is fairly obvious that a function, for which $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y$ *might* equal $6x - 5$, would be a function of the form

$$y = ax + b.$$

Substituting this "trial solution" into the equation, we get

$$0 - 2a - 3(ax + b) = 6x - 5,$$

or

$$(-3a)x + (-2a - 3b) = 6x - 5.$$

Equating coefficients of like terms we see that the equation is satisfied if

$$-3a = 6 \qquad \text{and} \qquad -2a - 3b = -5.$$

Solving these equations we have $a = -2$, $b = 3$; *i.e.*, the function

$$\text{(II)} \qquad y = -2x + 3$$

is a particular solution. It can be shown that the general solution can now be obtained by simply adding to this particular solution *the general solution of the same equation with the right-hand side replaced by zero.* The general solution of the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$ is

$$y = Ae^{-x} + Be^{3x}.$$

The general solution of (I) is then

$$\text{(III)} \qquad y = -2x + 3 + Ae^{-x} + Be^{3x}.$$

The proof consists merely in showing that (III) is a solution of (I) and noting that it contains the required number of arbitrary constants. The student will readily see that (III) must be a solution if (II) is, because the two extra terms in (III) give *zero* when substituted in the left side of (I).

This procedure can be used in solving any linear equation with constant coefficients. The function which is added to the particular integral to get the general solution is called the *complementary function.* It is always obtained by replacing the right-hand side of the given equation by zero and solving the resulting equation by the methods of the preceding article. A particular solution is often most easily found by the method of *undetermined coefficients* as demonstrated in the example just given. In using this method one "guesses" at the general form of a solution and substitutes this "trial solution" with undetermined coefficients into the equation. Then, if the trial solution has been properly selected, it is usually easy to determine the coefficients so that the equation will be satisfied.

No general rule for selecting the form of a trial solution can be given. In a case such as

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos x + 2 \sin x,$$

one must think of a type of function which *could have* the sum of its first and second derivatives equal to

$$\cos x + 2 \sin x.$$

A proper guess would of course be a function of the form

$$y = a \cos x + b \sin x.$$

If the right-hand side were $4e^{2x}$ instead of $\cos x + 2 \sin x$, a proper trial solution would of course be $y = ae^{2x}$.

PROBLEMS

1. What is a proper trial solution for the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = f(x),$$

if (a) $f(x) = 2x^2 + 3x + 4$, (b) $f(x) = 4 \sin x$, (c) $f(x) = e^{-x}$?

2. Why is $y = a \cos x$ not a proper trial solution for the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2 \cos x$? Find a particular solution and the general solution.

3. Explain why we should take $y = ax^3 + bx^2 + cx + d$ as a trial solution for the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^3$. Find a particular solution and the general solution.

4. Explain why $y = ax^3 + bx^2 + cx$ rather than $y = ax^2 + bx + c$ is a proper trial solution for the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 6x^2 + 2x + 3$. Find a particular solution and the general solution.

Solve the following equations:

5. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 2.$

6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3 \sin x.$

7. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6 \sin x + 8 \cos x.$

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 8y + 4x + 7.$

9. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4e^x.$

10. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 4x^2 + 8.$

11. $2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 3y = 9x^2.$

12. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x.$

13. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 8y = 2x + 7.$

14. $\dfrac{d^2y}{dx^2} = 3e^x + 4y + 8.$

15. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 10e^{2x}.$ HINT: If $y = ae^{2x}$ reduces the left side identically to zero, try $y = axe^{2x}$.

16. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 4y = 5e^{-x}.$ See hint in Prob. 15.

17. $\dfrac{d^2y}{dx^2} + y = 2 \sin x.$

18. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 12e^{3x}.$

19. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 2 \sin x.$

20. $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = 4x.$

21. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4.$

22. $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 6x^2.$

23. What is the equation of the curve which passes through (0, 2) at 45° and has at every point $d^2y/dx^2 = x + y$?

24. A chain 32 ft. long is hung over a smooth peg with 18 ft. of its length on one side. Using the relation $F = \dfrac{w}{g}a$ show that its acceleration is $\dfrac{d^2y}{dt^2} = g\left(\dfrac{y}{16} - 1\right)$ where y is the length of the longer side. Express y in terms of t.

25. The block shown in Fig. 171 weighs 80 lb. and rests on a smooth table. The springs are unstressed when the block is in the position shown. A force of 10 lb. per foot is required to stretch or compress either spring. Show that if the block is displaced a distance x to the right, the resultant force acting on it is $20x$ lb. to the left and its acceleration is

$$\frac{d^2x}{dt^2} = -\frac{gx}{4}.$$

Show that if the block is displaced 4 ft. to the right and released ($x = 4$ and $dx/dt = 0$ when $t = 0$), it will oscillate with respect to the position shown, the motion being described by the equation $x = 4 \cos \frac{\sqrt{g}}{2}t$. It is assumed that the weights of the springs may be neglected.

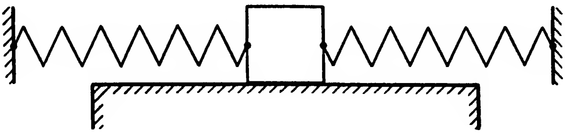

FIG. 171.

26. Solve Prob. 25 for the case in which the coefficient of friction between block and table is $\mu = 0.5$. Show in particular that in this case the block will not oscillate but will merely return to the original position and stop. Discuss the cases in which $\mu < 0.5$ and $\mu > 0.5$.

27. Discuss Prob. 25 for the case in which the motion takes place in a resisting medium in which the resistance is proportional to the velocity. Show that the differential equation is

$$\frac{80}{g}\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 20x = 0$$

where k is a positive constant. Note that in this case the motion will be oscillatory only for values of k for which the auxiliary equation has complex roots.

Table of Integrals

Forms containing $a + bx$

1. $$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x} + C.$$

2. $$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x} + C.$$

3. $$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x} + C.$$

4. $$\int \frac{x^2dx}{(a+bx)^2} = \frac{1}{b^3}\left[a + bx - \frac{a^2}{a+bx} - 2a\log(a+bx)\right] + C.$$

5. $$\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right] + C.$$

Forms containing $\sqrt{a + bx}$

6. $$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C.$$

7. $$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2 - 12abx + 15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C.$$

8. $$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)\sqrt{a+bx}}{3b^2} + C.$$

9. $$\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2 - 4abx + 3b^2x^2)\sqrt{a+bx}}{15b^3} + C.$$

Forms containing $\sqrt{x^2 + a^2}$

10. $$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2+a^2}) + C.$$

11. $$\int x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2+a^2}) + C.$$

12. $$\int \sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^4}{8}\log(x+\sqrt{x^2+a^2}) + C$$

13. $$\int \frac{\sqrt{x^2+a^2}}{x}dx = \sqrt{x^2+a^2} - a\log\frac{a+\sqrt{x^2+a^2}}{x} + C.$$

14. $$\int \frac{\sqrt{x^2+a^2}}{x^2}dx = -\frac{\sqrt{x^2+a^2}}{x} + \log(x+\sqrt{x^2+a^2}) + C.$$

15. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log(x+\sqrt{x^2+a^2}) + C.$$

16. $$\int \frac{x^2dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}) + C.$$

17. $$\int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C.$$

18. $\int \frac{x^2dx}{\sqrt{(x^2+a^2)^3}} = -\frac{x}{\sqrt{x^2+a^2}} + \log(x+\sqrt{x^2+a^2}) + C.$

19. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\log\frac{x}{a+\sqrt{x^2+a^2}} + C.$

20. $\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x} + C.$

Forms containing $\sqrt{x^2-a^2}$

21. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + C.$

22. $\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2-a^2}) + C.$

23. $\int \sqrt{(x^2-a^2)^3}\,dx = \frac{x}{8}(2x^2-5a^2)\sqrt{x^2-a^2} + \frac{3a^4}{8}\log(x+\sqrt{x^2-a^2}) + C.$

24. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\,\text{arc cos}\,\frac{a}{x} + C.$

25. $\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\frac{\sqrt{x^2-a^2}}{x} + \log(x+\sqrt{x^2-a^2}) + C.$

26. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}) + C.$

27. $\int \frac{x^2dx}{\sqrt{x^2-a^2}} = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + C.$

28. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C.$

29. $\int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C.$

Forms containing $\sqrt{a^2-x^2}$

30. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\text{arc sin}\,\frac{x}{a} + C.$

31. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\text{arc sin}\,\frac{x}{a} + C.$

32. $\int \frac{\sqrt{a^2-x^2}\,dx}{x} = \sqrt{a^2-x^2} - a\log\frac{a+\sqrt{a^2-x^2}}{x} + C.$

33. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - \text{arc sin}\,\frac{x}{a} + C.$

34. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C.$

35. $\int \frac{x^2dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\text{arc sin}\,\frac{x}{a} + C.$

36. $\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\log\frac{x}{a+\sqrt{a^2-x^2}} + C.$

37. $\displaystyle\int\frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C.$

38. $\displaystyle\int\frac{x^2dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \text{arc sin}\frac{x}{a} + C.$

Forms containing $\sqrt{2ax-x^2}$

39. $\displaystyle\int\sqrt{2ax-x^2}\,dx = \frac{(x-a)\sqrt{2ax-x^2}}{2} + \frac{a^2}{2}\text{arc sin}\frac{x-a}{a} + C.$

40. $\displaystyle\int\frac{dx}{\sqrt{2ax-x^2}} = \text{arc cos}\frac{a-x}{a} + C.$

41. $\displaystyle\int\frac{x\,dx}{\sqrt{2ax-x^2}} = -\sqrt{2ax-x^2} + a\,\text{arc cos}\frac{a-x}{a} + C.$

42. $\displaystyle\int\frac{x^2dx}{\sqrt{2ax-x^2}} = -\frac{(x+3a)\sqrt{2ax-x^2}}{2} + \frac{3a^2}{2}\text{arc cos}\frac{a-x}{a} + C.$

43. $\displaystyle\int\frac{dx}{x\sqrt{2ax-x^2}} = -\frac{\sqrt{2ax-x^2}}{ax} + C.$

44. $\displaystyle\int\frac{\sqrt{2ax-x^2}\,dx}{x} = \sqrt{2ax-x^2} + a\,\text{arc cos}\frac{a-x}{a} + C.$

45. $\displaystyle\int\frac{\sqrt{2ax-x^2}\,dx}{x^2} = -\frac{2\sqrt{2ax-x^2}}{x} - \text{arc cos}\frac{a-x}{a} + C.$

46. $\displaystyle\int\frac{\sqrt{2ax-x^2}\,dx}{x^3} = -\frac{(2ax-x^2)^{\frac{3}{2}}}{3ax^3} + C.$

Trigonometric forms

47. $\displaystyle\int\sin^2 x\,dx = \tfrac{1}{2}(x - \sin x\cos x) + C.$

48. $\displaystyle\int\cos^2 x\,dx = \tfrac{1}{2}(x + \sin x\cos x) + C.$

49. $\displaystyle\int\sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int\sin^{n-2}x\,dx.$

50. $\displaystyle\int\cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int\cos^{n-2}x\,dx.$

51. $$\begin{aligned}\int\sin^m x\cos^n x\,dx &= \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int\sin^m x\cos^{n-2}x\,dx\\ &= -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int\sin^{m-2}x\cos^n x\,dx.\end{aligned}$$

52. $\displaystyle\int\tan^2 x\,dx = \tan x - x + C.$

53. $\displaystyle\int\tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int\tan^{n-2}x\,dx.$

54. $\displaystyle\int\cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int\cot^{n-2}x\,dx.$

55. $\displaystyle\int\sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int\sec^{n-2}x\,dx.$

56. $\int \operatorname{arc\,sin} x \, dx = x \operatorname{arc\,sin} x + \sqrt{1 - x^2} + C.$

57. $\int \operatorname{arc\,cos} x \, dx = x \operatorname{arc\,cos} x - \sqrt{1 - x^2} + C.$

58. $\int \operatorname{arc\,tan} x \, dx = x \operatorname{arc\,tan} x - \frac{1}{2} \log (1 + x^2) + C.$

Miscellaneous forms

59. $\int x \sin x \, dx = \sin x - x \cos x + C.$

60. $\int x \cos x \, dx = \cos x + x \sin x + C.$

61. $\int e^{ax} \sin nx \, dx = \dfrac{e^{ax}(a \sin nx - n \cos nx)}{a^2 + n^2} + C.$

62. $\int e^{ax} \cos nx \, dx = \dfrac{e^{ax}(n \sin nx + a \cos nx)}{a^2 + n^2} + C.$

63. $\int x e^{ax} dx = \dfrac{e^{ax}(ax - 1)}{a^2} + C.$

64. $\int x^n e^{ax} dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1} e^{ax} dx.$

65. $\int x^n \log x \, dx = x^{n+1}\left[\dfrac{\log x}{n + 1} - \dfrac{1}{(n + 1)^2}\right] + C.$

Wallis' formulas

66. $\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

$= \dfrac{(n - 1)(n - 3) \cdots 4 \cdot 2}{n(n - 2) \cdots 5 \cdot 3 \cdot 1}$, if n is a positive odd integer > 1.

$= \dfrac{(n - 1)(n - 3) \cdots 3 \cdot 1}{n(n - 2) \cdots 4 \cdot 2} \cdot \dfrac{\pi}{2}$, if n is a positive even integer.

67. $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$

$= \dfrac{(n - 1)(n - 3) \cdots 4 \cdot 2}{(m + n)(m + n - 2) \cdots (m + 3)(m + 1)}$, if n is a positive odd integer > 1.

$= \dfrac{(m - 1)(m - 3) \cdots 4 \cdot 2}{(n + m)(n + m - 2) \cdots (n + 3)(n + 1)}$, if m is a positive odd integer > 1.

$= \dfrac{(m - 1)(m - 3) \cdots 3 \cdot 1 (n - 1)(n - 3) \cdots 3 \cdot 1}{(m + n)(m + n - 2) \cdots 4 \cdot 2} \cdot \dfrac{\pi}{2}$, if m and n are both positive even integers.

NUMERICAL TABLES

TABLE I.—COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

TABLE I.—COMMON LOGARITHMS.—(*Continued*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

TABLE II.—TRIGONOMETRIC FUNCTIONS

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
0° 00′	.0000	∞	1.0000	0.0000	.0000	∞	∞	∞	90° 00′
10	.0029	7.4637	1.0000	0000	.0029	7.4637	343.77	2.5363	50
20	.0058	7648	1.0000	0000	.0058	7648	171.89	2352	40
30	.0087	9408	1.0000	0000	.0087	9409	114.59	0591	30
40	.0116	8.0658	.9999	0000	.0116	8.0658	85.940	1.9342	20
50	.0145	1627	.9999	0000	.0145	1627	68.750	8373	10
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.290	1.7581	89° 00′
10	.0204	3088	.9998	9999	.0204	3089	49.104	6911	50
20	.0233	3668	.9997	9999	.0233	3669	42.964	6331	40
30	.0262	4179	.9997	9999	.0262	4181	38.188	5819	30
40	.0291	4637	.9996	9998	.0291	4638	34.368	5362	20
50	.0320	5050	.9995	9998	.0320	5053	31.242	4947	10
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.636	1.4569	88° 00′
10	.0378	5776	.9993	9997	.0378	5779	26.432	4221	50
20	.0407	6097	.9992	9996	.0407	6101	24.542	3899	40
30	.0436	6397	.9990	9996	.0437	6401	22.904	3599	30
40	.0465	6677	.9989	9995	.0466	6682	21.470	3318	20
50	.0494	6940	.9988	9995	.0495	6945	20.206	3055	10
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.081	1.2806	87° 00′
10	.0552	7423	.9985	9993	.0553	7429	18.075	2571	50
20	.0581	7645	.9983	9993	.0582	7652	17.169	2348	40
30	.0610	7857	.9981	9992	.0612	7865	16.350	2135	30
40	.0640	8059	.9980	9991	.0641	8067	15.605	1933	20
50	.0669	8251	.9978	9990	.0670	8261	14.924	1739	10
4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.301	1.1554	86° 00′
10	.0727	8613	.9974	9989	.0729	8624	13.727	1376	50
20	.0756	8783	.9971	9988	.0758	8795	13.197	1205	40
30	.0785	8946	.9969	9987	.0787	8960	12.706	1040	30
40	.0814	9104	.9967	9986	.0816	9118	12.251	0882	20
50	.0843	9256	.9964	9985	.0846	9272	11.826	0728	10
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.430	1.0580	85° 00′
10	.0901	9545	.9959	9982	.0904	9563	11.059	0437	50
20	.0929	9682	.9957	9981	.0934	9701	10.712	0299	40
30	.0958	9816	.9954	9980	.0963	9836	10.385	0164	30
40	.0987	9945	.9951	9979	.0992	9966	10.078	0034	20
50	.1016	9.0070	.9948	9977	.1022	9.0093	9.7882	0.9907	10
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.5144	0.9784	84° 00′
10	.1074	0311	.9942	9975	.1080	0336	9.2553	9664	50
20	.1103	0426	.9939	9973	.1110	0453	9.0098	9547	40
30	.1132	0539	.9936	9972	.1139	0567	8.7769	9433	30
40	.1161	0648	.9932	9971	.1169	0678	8.5555	9322	20
50	.1190	0755	.9929	9969	.1198	0786	8.3450	9214	10
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.1443	0.9109	83° 00′
10	.1248	0961	.9922	9966	.1257	0995	7.9530	9005	50
20	.1276	1060	.9918	9964	.1287	1096	7.7704	8904	40
30	.1305	1157	.9914	9963	.1317	1194	7.5958	8806	30
40	.1334	1252	.9911	9961	.1346	1291	7.4287	8709	20
50	.1363	1345	.9907	9959	.1376	1385	7.2687	8615	10
8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.1154	0.8522	82° 00′
10	.1421	1525	.9899	9956	.1435	1569	6.9682	8431	50
20	.1449	1612	.9894	9954	.1465	1658	6.8269	8342	40
30	.1478	1697	.9890	9952	.1495	1745	6.6912	8255	30
40	.1507	1781	.9886	9950	.1524	1831	6.5606	8169	20
50	.1536	1863	.9881	9948	.1554	1915	6.4348	8085	10
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

Table II.—Trigonometric Functions.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
10	.1593	2022	.9872	9944	.1614	2078	6.1970	7922	50
20	.1622	2100	.9868	9942	.1644	2158	6.0844	7842	40
30	.1650	2176	.9863	9940	.1673	2236	5.9758	7764	30
40	.1679	2251	.9858	9938	.1703	2313	5.8708	7687	20
50	.1708	2324	.9853	9936	.1733	2389	5.7694	7611	10
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.6713	0.7537	80° 00′
10	.1765	2468	.9843	9931	.1793	2536	5.5764	7464	50
20	.1794	2538	.9838	9929	.1823	2609	5.4845	7391	40
30	.1822	2606	.9833	9927	.1853	2680	5.3955	7320	30
40	.1851	2674	.9827	9924	.1883	2750	5.3093	7250	20
50	.1880	2740	.9822	9922	.1914	2819	5.2257	7181	10
11° 00′	.1908	9.2806	.9816	9.9919	.1944	9.2887	5.1446	0.7113	79° 00′
10	.1937	2870	.9811	9917	.1974	2953	5.0658	7047	50
20	.1965	2934	.9805	9914	.2004	3020	4.9894	6980	40
30	.1994	2997	.9799	9912	.2035	3085	4.9152	6915	30
40	.2022	3058	.9793	9909	.2065	3149	4.8430	6851	20
50	.2051	3119	.9787	9907	.2095	3212	4.7729	6788	10
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.7046	0.6725	78° 00′
10	.2108	3238	.9775	9901	.2156	3336	4.6382	6664	50
20	.2136	3296	.9769	9899	.2186	3397	4.5736	6603	40
30	.2164	3353	.9763	9896	.2217	3453	4.5107	6542	30
40	.2193	3410	.9757	9893	.2247	3517	4.4494	6483	20
50	.2221	3466	.9750	9890	.2278	3576	4.3897	6424	10
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.3315	0.6366	77° 00′
10	.2278	3575	.9737	9884	.2339	3691	4.2747	6309	50
20	.2306	3629	.9730	9881	.2370	3748	4.2193	6252	40
30	.2334	3682	.9724	9878	.2401	3804	4.1653	6196	30
40	.2363	3734	.9717	9875	.2432	3859	4.1126	6141	20
50	.2391	3786	.9710	9872	.2462	3914	4.0611	6086	10
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.0108	0.6032	76° 00′
10	.2447	3887	.9696	9866	.2524	4021	3.9617	5979	50
20	.2476	3937	.9689	9863	.2555	4074	3.9136	5926	40
30	.2504	3986	.9681	9859	.2586	4127	3.8667	5873	30
40	.2532	4035	.9674	9856	.2617	4178	3.8208	5822	20
50	.2560	4083	.9667	9853	.2648	4230	3.7760	5770	10
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.7321	0.5719	75° 00′
10	.2616	4177	.9652	9846	.2711	4331	3.6891	5669	50
20	.2644	4223	.9644	9843	.2742	4381	3.6470	5619	40
30	.2672	4269	.9636	9839	.2773	4430	3.6059	5570	30
40	.2700	4314	.9628	9836	.2805	4479	3.5656	5521	20
50	.2728	4359	.9621	9832	.2836	4527	3.5261	5473	10
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.4874	0.5425	74° 00′
10	.2784	4447	.9605	9825	.2899	4622	3.4495	5378	50
20	.2812	4491	.9596	9821	.2931	4669	3.4124	5331	40
30	.2840	4533	.9588	9817	.2962	4716	3.3759	5284	30
40	.2868	4576	.9580	9814	.2994	4762	3.3402	5238	20
50	.2896	4618	.9572	9810	.3026	4808	3.3052	5192	10
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.2709	0.5147	73° 00′
10	.2952	4700	.9555	9802	.3089	4898	3.2371	5102	50
20	.2979	4741	.9546	9798	.3121	4943	3.2041	5057	40
30	.3007	4781	.9537	9794	.3153	4987	3.1716	5013	30
40	.3035	4821	.9528	9790	.3185	5031	3.1397	4969	20
50	.3062	4861	.9520	9786	.3217	5075	3.1084	4925	10
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
10	.3118	4939	.9502	9778	.3281	5161	3.0475	4839	50
20	.3145	4977	.9492	9774	.3314	5203	3.0178	4797	40
30	.3173	5015	.9483	9770	.3346	5245	2.9887	4755	30
40	.3201	5052	.9474	9765	.3378	5287	2.9600	4713	20
50	.3228	5090	.9465	9761	.3411	5329	2.9319	4671	10
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.9042	0.4630	71° 00′
10	.3283	5163	.9446	9752	.3476	5411	2.8770	4589	50
20	.3311	5199	.9436	9748	.3508	5451	2.8502	4549	40
30	.3338	5235	.9426	9743	.3541	5491	2.8239	4509	30
40	.3365	5270	.9417	9739	.3574	5531	2.7980	4469	20
50	.3393	5306	.9407	9734	.3607	5571	2.7725	4429	10
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.7475	0.4389	70° 00′
10	.3448	5375	.9387	9725	.3673	5650	2.7228	4350	50
20	.3475	5409	.9377	9721	.3706	5689	2.6985	4311	40
30	.3502	5443	.9367	9716	.3739	5727	2.6746	4273	30
40	.3529	5477	.9356	9711	.3772	5766	2.6511	4234	20
50	.3557	5510	.9346	9706	.3805	5804	2.6279	4196	10
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.6051	0.4158	69° 00′
10	.3611	5576	.9325	9697	.3872	5879	2.5826	4121	50
20	.3638	5609	.9315	9692	.3906	5917	2.5605	4083	40
30	.3665	5641	.9304	9687	.3939	5954	2.5386	4046	30
40	.3692	5673	.9293	9682	.3973	5991	2.5172	4009	20
50	.3719	5704	.9283	9677	.4006	6028	2.4960	3972	10
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.4751	0.3936	68° 00′
10	.3773	5767	.9261	9667	.4074	6100	2.4545	3900	50
20	.3800	5798	.9250	9661	.4108	6136	2.4342	3864	40
30	.3827	5828	.9239	9656	.4142	6172	2.4142	3828	30
40	.3854	5859	.9228	9651	.4176	6208	2.3945	3792	20
50	.3881	5889	.9216	9646	.4210	6243	2.3750	3757	10
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.3559	0.3721	67° 00′
10	.3934	5948	.9194	9635	.4279	6314	2.3369	3686	50
20	.3961	5978	.9182	9629	.4314	6348	2.3183	3652	40
30	.3987	6007	.9171	9624	.4348	6383	2.2998	3617	30
40	.4014	6036	.9159	9618	.4383	6417	2.2817	3583	20
50	.4041	6065	.9147	9613	.4417	6452	2.2637	3548	10
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.2460	0.3514	66° 00′
10	.4094	6121	.9124	9602	.4487	6520	2.2286	3480	50
20	.4120	6149	.9112	9596	.4522	6553	2.2113	3447	40
30	.4147	6177	.9100	9590	.4557	6587	2.1943	3413	30
40	.4173	6205	.9088	9584	.4592	6620	2.1775	3380	20
50	.4200	6232	.9075	9579	.4628	6654	2.1609	3346	10
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.1445	0.3313	65° 00′
10	.4253	6286	.9051	9567	.4699	6720	2.1283	3280	50
20	.4279	6313	.9038	9561	.4734	6752	2.1123	3248	40
30	.4305	6340	.9026	9555	.4770	6785	2.0965	3215	30
40	.4331	6366	.9013	9549	.4806	6817	2.0809	3183	20
50	.4358	6392	.9001	9543	.4841	6850	2.0655	3150	10
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.0503	0.3118	64° 00′
10	.4410	6444	.8975	9530	.4913	6914	2.0353	3086	50
20	.4436	6470	.8962	9524	.4950	6946	2.0204	3054	40
30	.4462	6495	.8949	9518	.4986	6977	2.0057	3023	30
40	.4488	6521	.8936	9512	.5022	7009	1.9912	2991	20
50	.4514	6546	.8923	9505	.5059	7040	1.9768	2960	10
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
10	.4566	6595	.8897	9492	.5132	7103	1.9486	2897	50
20	.4592	6620	.8884	9486	.5169	7134	1.9347	2866	40
30	.4617	6644	.8870	9479	.5206	7165	1.9210	2835	30
40	.4643	6668	.8857	9473	.5243	7196	1.9074	2804	20
50	.4669	6692	.8843	9466	.5280	7226	1.8940	2774	10
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	62° 00′
10	.4720	6740	.8816	9453	.5354	7287	1.8676	2713	50
20	.4746	6763	.8802	9446	.5392	7317	1.8546	2683	40
30	.4772	6787	.8788	9439	.5430	7348	1.8418	2652	30
40	.4797	6810	.8774	9432	.5467	7378	1.8291	2622	20
50	.4823	6833	.8760	9425	.5505	7408	1.8165	2592	10
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	61° 00′
10	.4874	6878	.8732	9411	.5581	7467	1.7917	2533	50
20	.4899	6901	.8718	9404	.5619	7497	1.7796	2503	40
30	.4924	6923	.8704	9397	.5658	7526	1.7675	2474	30
40	.4950	6946	.8689	9390	.5696	7556	1.7556	2444	20
50	.4975	6968	.8675	9383	.5735	7585	1.7437	2415	10
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	60° 00′
10	.5025	7012	.8646	9368	.5812	7644	1.7205	2356	50
20	.5050	7033	.8631	9361	.5851	7673	1.7090	2327	40
30	.5075	7055	.8616	9353	.5890	7701	1.6977	2299	30
40	.5100	7076	.8601	9346	.5930	7730	1.6864	2270	20
50	.5125	7097	.8587	9338	.5969	7759	1.6753	2241	10
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	59° 00′
10	.5175	7139	.8557	9323	.6048	7816	1.6534	2184	50
20	.5200	7160	.8542	9315	.6088	7845	1.6426	2155	40
30	.5225	7181	.8526	9308	.6128	7873	1.6319	2127	30
40	.5250	7201	.8511	9300	.6168	7902	1.6212	2098	20
50	.5275	7222	.8496	9292	.6208	7930	1.6107	2070	10
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	58° 00′
10	.5324	7262	.8465	9276	.6289	7986	1.5900	2014	50
20	.5348	7282	.8450	9268	.6330	8014	1.5798	1986	40
30	.5373	7302	.8434	9260	.6371	8042	1.5697	1958	30
40	.5398	7322	.8418	9252	.6412	8070	1.5597	1930	20
50	.5422	7342	.8403	9244	.6453	8097	1.5497	1903	10
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	57° 00′
10	.5471	7380	.8371	9228	.6536	8153	1.5301	1847	50
20	.5495	7400	.8355	9219	.6577	8180	1.5204	1820	40
30	.5519	7419	.8339	9211	.6619	8208	1.5108	1792	30
40	.5544	7438	.8323	9203	.6661	8235	1.5013	1765	20
50	.5568	7457	.8307	9194	.6703	8263	1.4919	1737	10
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	56° 00′
10	.5616	7494	.8274	9177	.6787	8317	1.4733	1683	50
20	.5640	7513	.8258	9169	.6830	8344	1.4641	1656	40
30	.5664	7531	.8241	9160	.6873	8371	1.4550	1629	30
40	.5688	7550	.8225	9151	.6916	8398	1.4460	1602	20
50	.5712	7568	.8208	9142	.6959	8425	1.4370	1575	10
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	55° 00′
10	.5760	7604	.8175	9125	.7046	8479	1.4193	1521	50
20	.5783	7622	.8158	9116	.7089	8506	1.4106	1494	40
30	.5807	7640	.8141	9107	.7133	8533	1.4019	1467	30
40	.5831	7657	.8124	9098	.7177	8559	1.3934	1441	20
50	.5854	7675	.8107	9089	.7221	8586	1.3848	1414	10
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS.—(*Continued*)

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
10	.5901	7710	.8073	9070	.7310	8639	1.3680	1361	50
20	.5925	7727	.8056	9061	.7355	8666	1.3597	1334	40
30	.5948	7744	.8039	9052	.7400	8692	1.3514	1308	30
40	.5972	7761	.8021	9042	.7445	8718	1.3432	1282	20
50	.5995	7778	.8004	9033	.7490	8745	1.3351	1255	10
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.3270	0.1229	53° 00′
10	.6041	7811	.7969	9014	.7581	8797	1.3190	1203	50
20	.6065	7828	.7951	9004	.7627	8824	1.3111	1176	40
30	.6088	7844	.7934	8995	.7673	8850	1.3032	1150	30
40	.6111	7861	.7916	8985	.7720	8876	1.2954	1124	20
50	.6134	7877	.7898	8975	.7766	8902	1.2876	1098	10
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.2790	0.1072	52° 00′
10	.6180	7910	.7862	8955	.7860	8954	1.2723	1046	50
20	.6202	7926	.7844	8945	.7907	8980	1.2647	1020	40
30	.6225	7941	.7826	8935	.7954	9006	1.2572	0994	30
40	.6248	7957	.7808	8925	.8002	9032	1.2497	0968	20
50	.6271	7973	.7790	8915	.8050	9058	1.2423	0942	10
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.2349	0.0916	51° 00′
10	.6316	8004	.7753	8895	.8146	9110	1.2276	0890	50
20	.6338	8020	.7735	8884	.8195	9135	1.2203	0865	40
30	.6361	8035	.7716	8874	.8243	9161	1.2131	0839	30
40	.6383	8050	.7698	8864	.8292	9187	1.2059	0813	20
50	.6406	8066	.7679	8853	.8342	9212	1.1988	0788	10
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.1918	0.0762	50° 00′
10	.6450	8096	.7642	8832	.8441	9264	1.1847	0736	50
20	.6472	8111	.7623	8821	.8491	9289	1.1778	0711	40
30	.6494	8125	.7604	8810	.8541	9315	1.1708	0685	30
40	.6517	8140	.7585	8800	.8591	9341	1.1640	0659	20
50	.6539	8155	.7566	8789	.8642	9366	1.1571	0634	10
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.1504	0.0608	49° 00′
10	.6583	8184	.7528	8767	.8744	9417	1.1436	0583	50
20	.6604	8198	.7509	8756	.8796	9443	1.1369	0557	40
30	.6626	8213	.7490	8745	.8847	9468	1.1303	0532	30
40	.6648	8227	.7470	8733	.8899	9494	1.1237	0506	20
50	.6670	8241	.7451	8722	.8952	9519	1.1171	0481	10
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.1106	0.0456	48° 00′
10	.6713	8269	.7412	8699	.9057	9570	1.1041	0430	50
20	.6734	8283	.7392	8688	.9110	9595	1.0977	0405	40
30	.6756	8297	.7373	8676	.9163	9621	1.0913	0379	30
40	.6777	8311	.7353	8665	.9217	9646	1.0850	0354	20
50	.6799	8324	.7333	8653	.9271	9671	1.0786	0329	10
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.0724	0.0303	47° 00′
10	.6841	8351	.7294	8629	.9380	9722	1.0661	0278	50
20	.6862	8365	.7274	8618	.9435	9747	1.0599	0253	40
30	.6884	8378	.7254	8606	.9490	9772	1.0538	0228	30
40	.6905	8391	.7234	8594	.9545	9798	1.0477	0202	20
50	.6926	8405	.7214	8582	.9601	9823	1.0416	0177	10
44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.0355	0.0152	46° 00′
10	.6967	8431	.7173	8557	.9713	9874	1.0295	0126	50
20	.6988	8444	.7153	8545	.9770	9899	1.0235	0101	40
30	.7009	8457	.7133	8532	.9827	9924	1.0176	0076	30
40	.7030	8469	.7112	8520	.9884	9949	1.0117	0051	20
50	.7050	8482	.7092	8507	.9942	9975	1.0058	0025	10
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	45° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE III.—POWERS AND ROOTS

No.	Sq.	Sq. Root	Cube	Cube Root	No.	Sq.	Sq. Root	Cube	Cube Root
1	1	1.000	1	1.000	**51**	2,601	7.141	132,651	3.708
2	4	1.414	8	1.260	**52**	2,704	7.211	140,608	3.733
3	9	1.732	27	1.442	**53**	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	**54**	2,916	7.348	157,464	3.780
5	25	2.236	125	1.710	**55**	3,025	7.416	166,375	3.803
6	36	2.449	216	1.817	**56**	3,136	7.483	175,616	3.826
7	49	2.646	343	1.913	**57**	3,249	7.550	185,193	3.849
8	64	2.828	512	2.000	**58**	3,364	7.616	195,112	3.871
9	81	3.000	729	2.080	**59**	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	**60**	3,600	7.746	216,000	3.915
11	121	3.317	1,331	2.224	**61**	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	**62**	3,844	7.874	238,328	3.958
13	169	3.606	2,197	2.351	**63**	3,969	7.937	250,047	3.979
14	196	3.742	2,744	2.410	**64**	4,096	8.000	262,144	4.000
15	225	3.873	3,375	2.466	**65**	4,225	8.062	274,625	4.021
16	256	4.000	4,096	2.520	**66**	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	**67**	4,489	8.185	300,763	4.062
18	324	4.243	5,832	2.621	**68**	4,624	8.246	314,432	4.082
19	361	4.359	6,859	2.668	**69**	4,761	8.307	328,509	4.102
20	400	4.472	8,000	2.714	**70**	4,900	8.367	343,000	4.121
21	441	4.583	9,261	2.759	**71**	5,041	8.426	357,911	4.141
22	484	4.690	10,648	2.802	**72**	5,184	8.485	373,248	4.160
23	529	4.796	12,167	2.844	**73**	5,329	8.544	389,017	4.179
24	576	4.899	13,824	2.884	**74**	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	**75**	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	**76**	5,776	8.718	438,976	4.236
27	729	5.196	19,683	3.000	**77**	5,929	8.775	456,533	4.254
28	784	5.291	21,952	3.037	**78**	6,084	8.832	474,552	4.273
29	841	5.385	24,389	3.072	**79**	6,241	8.888	493,039	4.291
30	900	5.477	27,000	3.107	**80**	6,400	8.944	512,000	4.309
31	961	5.568	29,791	3.141	**81**	6,561	9.000	531,441	4.327
32	1,024	5.657	32,768	3.175	**82**	6,724	9.055	551,368	4.344
33	1,089	5.745	35,937	3.208	**83**	6,889	9.110	571,787	4.362
34	1,156	5.831	39,304	3.240	**84**	7,056	9.165	592,704	4.380
35	1,225	5.916	42,875	3.271	**85**	7,225	9.220	614,125	4.397
36	1,296	6.000	46,656	3.302	**86**	7,396	9.274	636,056	4.414
37	1,369	6.083	50,653	3.332	**87**	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.362	**88**	7,744	9.381	681,472	4.448
39	1,521	6.245	59,319	3.391	**89**	7,921	9.434	704,969	4.465
40	1,600	6.325	64,000	3.420	**90**	8,100	9.487	729,000	4.481
41	1,681	6.403	68,921	3.448	**91**	8,281	9.539	753,571	4.498
42	1,764	6.481	74,088	3.476	**92**	8,464	9.592	778,688	4.514
43	1,849	6.557	79,507	3.503	**93**	8,649	9.644	804,357	4.531
44	1,936	6.633	85,184	3.530	**94**	8,836	9.695	830,584	4.547
45	2,025	6.708	91,125	3.557	**95**	9,025	9.747	857,375	4.563
46	2,116	6.782	97,336	3.583	**96**	9,216	9.798	884,736	4.579
47	2,209	6.856	103,823	3.609	**97**	9,409	9.849	912,673	4.595
48	2,304	6.928	110,592	3.634	**98**	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	**99**	9,801	9.950	970,299	4.626
50	2,500	7.071	125,000	3.684	**100**	10,000	10.000	1,000,000	4.642

TABLE IV.—NATURAL LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0 000	100	198	296	392	488	583	677	700	862
1.1	953	*044	*133	*222	*310	*398	*484	*570	*655	*740
1.2	0.1 823	906	989	*070	*151	*231	*311	*390	*469	*546
1.3	0.2 624	700	776	852	927	*001	*075	*148	*221	*293
1.4	0.3 365	436	507	577	646	716	784	853	920	988
1.5	0.4 055	121	187	253	318	383	447	511	574	637
1.6	700	762	824	886	947	*008	*068	*128	*188	*247
1.7	0.5 306	365	423	481	539	596	653	710	766	822
1.8	878	933	988	*043	*098	*152	*206	*259	*313	*366
1.9	0.6 419	471	523	575	627	678	729	780	831	881
2.0	931	981	*031	*080	*129	*178	*227	*275	*324	*372
2.1	0.7 419	467	514	561	608	655	701	747	793	839
2.2	885	930	975	*020	*065	*109	*154	*198	*242	*286
2.3	0.8 329	372	416	459	502	544	587	629	671	713
2.4	755	796	838	879	920	961	*002	*042	*083	*123
2.5	0.9 163	203	243	282	322	361	400	439	478	517
2.6	555	594	632	670	708	746	783	821	858	895
2.7	933	969	*006	*043	*080	*116	*152	*188	*225	*260
2.8	1.0 296	332	367	403	438	473	508	543	578	613
2.9	647	682	716	750	784	818	852	886	919	953
3.0	986	*019	*053	*086	*119	*151	*184	*217	*249	*282
3.1	1.1 314	346	378	410	442	474	506	537	569	600
3.2	632	663	694	725	756	787	817	848	878	909
3.3	939	969	*000	*030	*060	*090	*119	*149	*179	*208
3.4	1.2 238	267	296	326	355	384	413	442	470	499
3.5	528	556	585	613	641	669	698	726	754	782
3.6	809	837	865	892	920	947	975	*002	*029	*056
3.7	1.3 083	110	137	164	191	218	244	271	297	324
3.8	350	376	402	429	455	481	507	533	558	584
3.9	610	635	661	686	712	737	762	788	813	838
4.0	863	888	913	938	962	987	*012	*036	*061	*085
4.1	1.4 110	134	159	183	207	231	255	279	303	327
4.2	351	375	398	422	446	469	493	516	540	563
4.3	586	609	633	656	679	702	725	748	770	793
4.4	816	839	861	884	907	929	951	974	996	*019
4.5	1.5 041	063	085	107	129	151	173	195	217	239
4.6	261	282	304	326	347	369	390	412	433	454
4.7	476	497	518	539	560	581	602	623	644	655
4.8	686	707	728	748	769	790	810	831	851	872
4.9	892	913	933	953	974	994	*014	*034	*054	*074
5.0	1.6 094	114	134	154	174	194	214	233	253	273

If given number $n = N \times 10^m$, then $\log_e n = \log_e N + m \log_e 10$. Find $m \log_e 10$ from the following table:

Multiples of $\log_e 10$

$\log_e 10 = 2.3026$	$-\log_e 10 = 7.6974 - 10$
$2 \log_e 10 = 4.6052$	$-2 \log_e 10 = 5.3948 - 10$
$3 \log_e 10 = 6.9078$	$-3 \log_e 10 = 3.0922 - 10$
$4 \log_e 10 = 9.2103$	$-4 \log_e 10 = 0.7897 - 10$
$5 \log_e 10 = 11.5129$	$-5 \log_e 10 = 9.4871 - 20$

TABLE IV.—NATURAL LOGARITHMS.—(*Continued*)

N	0	1	2	3	4	5	6	7	8	9
5.0	1.6 094	114	134	154	174	194	214	233	253	273
5.1	292	312	332	351	371	390	409	429	448	467
5.2	487	506	525	544	563	582	601	620	639	658
5.3	677	696	715	734	752	771	790	808	827	845
5.4	864	882	901	919	938	956	974	993	*011	*029
5.5	1.7 047	066	084	102	120	138	156	174	192	210
5.6	228	246	263	281	299	317	334	352	370	387
5.7	405	422	440	457	475	492	509	527	544	561
5.8	579	596	613	630	647	664	681	699	716	733
5.9	750	766	783	800	817	834	851	867	884	901
6.0	918	934	951	967	984	*001	*017	*034	*050	*066
6.1	1.8 083	099	116	132	148	165	181	197	213	229
6.2	245	262	278	294	310	326	342	358	374	390
6.3	405	421	437	453	469	485	500	516	532	547
6.4	563	579	594	610	625	641	656	672	687	703
6.5	718	733	749	764	779	795	810	825	840	856
6.6	871	886	901	916	931	946	961	976	991	*006
6.7	1.9 021	036	051	066	081	095	110	125	140	155
6.8	169	184	199	213	228	242	257	272	286	301
6.9	315	330	344	359	373	387	402	416	430	445
7.0	459	473	488	502	516	530	544	559	573	587
7.1	601	615	629	643	657	671	685	699	713	727
7.2	741	755	769	782	796	810	824	838	851	865
7.3	879	892	906	920	933	947	961	974	988	*001
7.4	2.0 015	028	042	055	069	082	096	109	122	136
7.5	149	162	176	189	202	215	229	242	255	268
7.6	281	295	308	321	334	347	360	373	386	399
7.7	412	425	438	451	464	477	490	503	516	528
7.8	541	554	567	580	592	605	618	631	643	656
7.9	669	681	694	707	719	732	744	757	769	782
8.0	794	807	819	832	844	857	869	882	894	906
8.1	919	931	943	956	968	980	992	*005	*017	*029
8.2	2.1 041	054	066	080	090	102	114	126	138	150
8.3	163	175	187	199	211	223	235	247	258	270
8.4	282	294	306	318	330	342	353	365	377	389
8.5	401	412	424	436	448	460	471	483	494	506
8.6	518	529	541	552	564	576	587	599	610	622
8.7	633	645	656	668	679	691	702	713	725	736
8.8	748	759	770	782	793	804	815	827	838	849
8.9	861	872	883	894	905	917	928	939	950	961
9.0	972	983	994	*006	*017	*028	*039	*050	*061	*072
9.1	2.2 083	094	105	116	127	137	148	159	170	181
9.2	192	203	214	225	235	246	257	268	279	289
9.3	300	311	322	332	343	354	364	375	386	396
9.4	407	418	428	439	450	460	471	481	492	502
9.5	513	523	534	544	555	565	576	586	597	607
9.6	618	628	638	649	659	670	680	690	701	711
9.7	721	732	742	752	762	773	783	793	803	814
9.8	824	834	844	854	865	875	885	895	905	915
9.9	925	935	946	956	966	976	986	996	*006	*016
10.	2.3 026	036	046	056	066	076	086	096	106	115

TABLE V.—EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	sinh x	cosh x	tanh x
.00	1.000	1.000	.000	1.000	.000
.01	1.010	.990	.010	1.000	.010
.02	1.020	.980	.020	1.000	.020
.03	1.030	.970	.030	1.000	.030
.04	1.041	.961	.040	1.001	.040
.05	1.051	.951	.050	1.001	.050
.06	1.062	.942	.060	1.002	.060
.07	1.073	.932	.070	1.002	.070
.08	1.083	.923	.080	1.003	.080
.09	1.094	.914	.090	1.004	.090
.1	1.105	.905	.100	1.005	.100
.2	1.221	.819	.201	1.020	.197
.3	1.350	.741	.305	1.045	.291
.4	1.492	.670	.411	1.081	.380
.5	1.649	.607	.521	1.128	.462
.6	1.822	.549	.637	1.185	.537
.7	2.014	.497	.759	1.255	.604
.8	2.226	.449	.888	1.337	.664
.9	2.460	.407	1.027	1.433	.716
1.0	2.718	.368	1.175	1.543	.762
1.1	3.004	.333	1.336	1.669	.800
1.2	3.320	.301	1.509	1.811	.834
1.3	3.669	.273	1.698	1.971	.862
1.4	4.055	.247	1.904	2.151	.885
1.5	4.481	.223	2.129	2.352	.905
1.6	4.953	.202	2.376	2.577	.922
1.7	5.474	.183	2.646	2.828	.935
1.8	6.050	.165	2.942	3.107	.947
1.9	6.686	.150	3.268	3.418	.956
2.0	7.389	.135	3.627	3.762	.964
2.1	8.166	.122	4.022	4.144	.970
2.2	9.025	.111	4.457	4.568	.976
2.3	9.974	.100	4.937	5.037	.980
2.4	11.023	.091	5.466	5.557	.984
2.5	12.182	.082	6.050	6.132	.987
2.6	13.464	.074	6.695	6.769	.989
2.7	14.880	.067	7.406	7.473	.991
2.8	16.444	.061	8.192	8.253	.993
2.9	18.174	.055	9.060	9.115	.994
3.0	20.086	.050	10.018	10.068	.995
3.1	22.20	.045	11.08	11.12	.996
3.2	24.53	.041	12.25	12.29	.997
3.3	27.11	.037	13.54	13.57	.997
3.4	29.96	.033	14.97	15.00	.998
3.5	33.12	.030	16.54	16.57	.998
3.6	36.60	.027	18.29	18.31	.999
3.7	40.45	.025	20.21	20.24	.999
3.8	44.70	.022	22.34	22.36	.999
3.9	49.40	.020	24.69	24.71	.999
4.0	54.60	.018	27.29	27.31	.999
4.1	60.34	.017	30.16	30.18	.999
4.2	66.69	.015	33.34	33.35	1.000
4.3	73.70	.014	36.84	36.86	1.000
4.4	81.45	.012	40.72	40.73	1.000
4.5	90.02	.011	45.00	45.01	1.000
4.6	99.48	.010	49.74	49.75	1.000
4.7	109.95	.0090	54.97	54.98	1.000
4.8	121.51	.0082	60.75	60.76	1.000
4.9	134.29	.0074	67.14	67.15	1.000
5.0	148.41	.0067	74.20	74.21	1.000
6.0	403.4	.0024	201.7		1.000
7.0	1096.6	.00091	548.3		1.000
8.0	2981.0	.00034	1490.5		1.000
9.0	8103.1	.00012	4051.5		1.000
10.0	22026.5	.000045-	11013.2		1.000

INDEX

ANSWERS

CHAPTER II

Art. 12, pp. 19–20

2. $x > \sqrt{500}$.

5. $P = 2\,nr \sin \dfrac{\pi}{n}$.

6. $A = nr^2 \tan \dfrac{\pi}{n}$.

8. 3.

Art. 14, pp. 23–24

1. 6. **2.** 8. **3.** 12. **4.** 18.

5. $\frac{3}{5}$. **6.** No limit. **10.** $\frac{1}{2}$. **12.** 3.

14. $P = 2\,nr \sin \dfrac{\pi}{n}$.

15. $A = nr^2 \sin \dfrac{\pi}{n} \cos \dfrac{\pi}{n}$.

16. 7. **17.** 2. **18.** $\frac{27}{2}$. **19.** $\frac{7}{11}$.

20. $|r| < 1$; $\dfrac{a}{1 - r}$.

CHAPTER III

Art. 17, pp. 29–30

3. $2x$. **4.** $2x - 7$. **5.** $4x - 5$. **6.** $8t - 8$.

7. a. **8.** $2ax + b$. **9.** $-\dfrac{1}{x^2}$. **10.** $-\dfrac{2}{x^3}$.

11. $\dfrac{-2x}{(x^2 - 1)^2}$. **12.** $\dfrac{-12x}{(x^2 + 1)^2}$. **13.** $\dfrac{6}{(x + 3)^2}$. **14.** $\dfrac{-3}{(x - 1)^2}$.

15. $3x^2$. **16.** $3x^2 - 16$. **17.** $3(t - 2)^2$. **18.** $3x^2 - 6x - 6$.

19. $4x^3$. **20.** $3x^2 - \dfrac{2}{x^3}$. **21.** 6. **22.** 0.

23. -1. **24.** 1. **25.** -9. **26.** 0.

27. 6 units per minute.

28. $\begin{cases} 18 \text{ units per minute.} \\ -6 \text{ units per minute.} \end{cases}$

29. $\dfrac{dy}{dx} = 4 - 2x$.

30. 16π cu. in. per inch.
8π cu. in. per minute.

32. $12x$ sq. in. per inch.
96 sq. in. per minute.

33. $\dfrac{dv}{dx} = 3x^2$, $\dfrac{ds}{dx} = 12x$.

CHAPTER IV

Art. 19, pp. 38–40

2. $3x^2 + 4$. **4.** $6x(x^2 + 1)^2$. **6.** $4x - 7$. **7.** $3x^2 - 6$.

8. $12t^2 - 8$. **9.** $4(2t + 1)$. **10.** $-\dfrac{x}{\sqrt{a^2 - x^2}}$. **11.** $\dfrac{1}{\sqrt{x}}$.

12. $-\dfrac{1}{2x^{\frac{3}{2}}}$ **13.** $6\sqrt{x}$. **14.** $\frac{3}{2}(2 - 3y)^{-\frac{3}{2}}$ **15.** $\dfrac{3x^4 - 32}{x^2}$.

16. $-\dfrac{4}{x^2}$. **17.** $\dfrac{x^2 - 2x - 5}{(x - 1)^2}$. **18.** $\dfrac{-2t}{(t^2 + 4)^2}$.

19. $\dfrac{2x^2 + 4}{\sqrt{x^2 + 4}}$. **20.** $-\dfrac{1}{2x^2\sqrt{2 + \frac{1}{x}}}$. **21.** $\dfrac{6x + 3}{\sqrt{4x - 1}}$.

22. $\dfrac{2x}{(1 + x^2)^{\frac{1}{2}}(1 - x^2)^{\frac{3}{2}}}$. **23.** $\dfrac{1}{(1 - 4y^2)^{\frac{3}{2}}}$.

24. $-\dfrac{bx}{a\sqrt{a^2 - x^2}}$. **25.** $1 - \sqrt{\dfrac{a}{x}}$. **26.** $-\dfrac{16a^3x}{(x^2 + 4a^2)^2}$.

27. $\dfrac{32x(x^2 - 4)}{(x^2 + 4)^3}$. **28.** $\dfrac{4(4 - t^2)}{(t^2 + 4)^2}$. **29.** $\dfrac{1}{(1 - x)^2} - \dfrac{4}{x^2}$.

30. 2. **31.** $-\dfrac{a}{b}$. **32.** $\pm\dfrac{1}{2^{\frac{3}{2}}\sqrt{x + 4}}$.

33. $\pm\dfrac{x}{\sqrt{r^2 - x^2}}$. **34.** $\pm\sqrt{\dfrac{p}{x}}$. **35.** $-\dfrac{16}{x^2}$.

36. $\pm\dfrac{bx}{a\sqrt{a^2 - x^2}}$. **37.** $x + 2$. **38.** -2.

39. -2. **40.** $-\frac{3}{4}$. **41.** $1; -1$. **42.** $\dfrac{3}{\sqrt{7}}$.

43. $-\dfrac{\sqrt{3}}{4}$. **45.** $3x^2 - 6x + 2$. **46.** $8x^3 - 9x^2 + 12x - 9$.

47. $16x^3 - 21x^2 - 28x + 21$. **50.** $\dfrac{dv}{dr} = 4\pi r^2$ cu. in. per inch.

$\dfrac{ds}{dr} = 8\pi r$ sq. in. per inch.

51. $\dfrac{dv}{ds} = \dfrac{r}{2}$ cu. in. per square inch. **52.** $\dfrac{\sqrt{3}x}{2}$ sq. ft. per foot.

53. $\dfrac{dv}{dx} = 3x^2$ cu. in. per inch.

$\dfrac{ds}{dx} = 12x$ sq. in. per inch.

Art. 21, pp. 43–44

1. $4(2x - 5)$. **2.** $2x(2x^2 - 1)$. **3.** $\dfrac{2x}{(1 - x^2)^2}$. **4.** $-\dfrac{2}{x^2}$.

5. $-\dfrac{1}{t^2}$. **6.** $\dfrac{-2}{(1 + t)^2}$. **7.** $\dfrac{8t(1 - t^4)}{(1 + t^4)^2}$.

8. $\dfrac{3(t^2 + 4t - 8)}{8}$. **9.** 6 units per unit of time.

10. $-4, 0, 12$, units per unit of time. **11.** $\dfrac{dy}{dt} = +3$.

14. $\frac{1}{18}; -\frac{1}{6}$. **15.** $-\frac{1}{6}; -\frac{3}{2}$. **16.** $r = 2$ in.

Art. 22, p. 45

1. $-\frac{3}{8}$.
2. $-\frac{a}{b}$.
3. $\frac{x}{y}$.
4. $\frac{8}{y}$.
5. $-\frac{x}{y}$.
6. $\frac{2x}{3y^2}$.
7. $-\frac{b^2x}{a^2y}$.
8. $-\sqrt{\frac{y}{x}}$.
9. $\frac{2x + y}{8y - x}$.
10. $\frac{4 - x}{y - 12}$.
11. $\frac{1}{y}$.
12. $\frac{-x}{4y + 2}$.
13. $-\frac{2x + y}{2y + x}$.
14. $\frac{3y + 4x}{2y - 3x}$.
15. $\frac{4 - 6x - 8y}{8x - 6y - 1}$.
16. $\frac{ay - x^2}{y^2 - ax}$.
17. $-\frac{4y^2 + 3x^2}{12y^2 + 8xy}$.
18. $\frac{x^2 - 2y}{4y^2 + 2x}$.
19. $\frac{8xy - y^2}{2xy - 4x^2 - 6y^2}$.
20. $\frac{2x^3 - xy^2}{4y^3 + x^2y}$.

CHAPTER V

Art. 24, pp. 43–49

1. $(-5, 0)$; $(3, 4)$; $\tan\varphi = \pm 2$.
2. $(1, \pm\sqrt{3})$; $\tan\varphi = \pm\sqrt{3}$.
3. At $(0, 0)$; $\varphi = 90°$.
   At $(1, 1)$; $\tan\varphi = \pm\frac{3}{4}$.
4. $\tan\varphi = \pm\frac{15}{8}$.
6. $\tan\varphi = \pm\frac{9}{7}$.
7. $\tan\varphi = \pm\frac{5}{3}$ at $(1, 0)$.
   $\tan\varphi = \pm\frac{5}{7}$ at $(6, 5)$.
8. Tan. $4x + 3y = 24$.
   Norm. $4y - 3x = 7$.
9. Tan. $3y - x = 6$.
   Norm. $y + 3x = 2$.
10. Tan. $y = 3x$.
    Norm. $3y + x = 10$.
11. Tan. $8y + 3x = 25$.
    Norm. $8x - 3y = 18$.
12. Tan. $6x - y = 24$.
    Norm. $6y + x = 4$.
    Tan. $y = -9$.
    Norm. $x = 1$.
13. $6x - y = 30$.
14. $3x + y + 10 = 0$.
15. $2y - x + 17 = 0$.
18. $\tan\varphi = \pm\frac{9}{17}$.

Art. 27, pp. 52–54

1. (*a*) 55.6 ft. per second.
   -8.8 ft. per second.
   (*b*) 223.6 ft.
   (*c*) 7.4 sec.
2. 197 ft. per second.
3. 205 ft. per second.
4. $x = 5$ units.
   $v = \frac{1}{6}$ unit per minute.
5. 8.9 ft. per second.
6. $\frac{4}{5}$ m.p.h.
7. $A = 2(12 + 4t - t^2)$.
9. 32 sq. in. per minute.
10. 400π cu. in. per minute.
11. $7\frac{1}{2}$ m.p.h.
    $2\frac{1}{2}$ m.p.h
12. 8 units per minute.
13. $5\sqrt{2}$ units per minute.
14. $\frac{4}{9\pi}$ in. per minute.

15. 5 cu. ft. per minute.

16. $\frac{8\sqrt{3}}{3}$ ft. per second.

17. $\frac{5\sqrt{420}}{13}$ ft. per second.

18. About 3 cu. ft. per minute.

Art. 28, pp. 56–57

4. max. $(2, 16)$, $(-2, 16)$. min. $(0, 0)$.

5. max. $(1, 5)$. min. $(3, 1)$.

6. min. $(5, -3)$. max. $(-1, 1)$.

7. min. $(2, -60)$. max. $(-2, 68)$. inf. $(0, 4)$.

8. min. $(2, 24)$.

9. min. $(2, 32)$. max. $(-2, -32)$.

10. max. $(2, 4)$. min. $(\frac{2}{3}, 36)$.

11. max. $(0, 4)$.

12. min. $(0, 0)$. max. $(-4, -8)$.

13. max. $(2, 1)$. min. $(-2, -1)$.

14. min. $(6, 5)$. max. $(2, -3)$.

15. max. $(0, b)$.

16. min. $(2, 0)$, $(-3, 0)$. max. $(-\frac{1}{2}, \frac{625}{16})$.

17. No horizontal tangent.

18. No max. or min.

19. min. $(8, 18)$. max. $(-2, -2)$.

20. min. $(-3, \frac{27}{2})$. inf. $(0, 0)$.

21. max. $(2\sqrt{2}, 8)$. min. $(-2\sqrt{2}, -8)$.

Art. 30, pp. 59–61

1. $2\frac{1}{4}$ cu. ft.

2. $6'$ by $6'$ by $3'$.

3. 6, 6.

5. $16\frac{1}{2}$ by 22 rods.

6. $4\frac{1}{2}$ miles from A.

8. $r = \frac{2}{3}a$, $h = \frac{1}{3}b$.

9. $t = \frac{6}{5}$ hr.

10. $\frac{4\pi r^3}{3\sqrt{3}}$.

11. 1:2.

14. $8\sqrt{2}$.

15. \$50 per mo.

16. \$52 per mo.

17. \$3.25 per mo.

18. \$3.50 per mo.

19. 4 ft. from A.

20. 25 by $16\frac{2}{3}$ yd.

21. $\frac{h}{r} = 5$.

22. radius of semicircle equals height of rectangle.

23. Width $= \frac{2r}{\sqrt{3}}$.

24. 2 ft.

25. $\frac{3\sqrt{3}r^2}{4}$.

26. 30 by 50 ft.

27. $\frac{3\sqrt{3}a}{4}$.

28. $9x^2 + 16y^2 = 288$.

CHAPTER VI

Art. 32, p. 64

1. $y'' = 6$.

2. $y'' = 6x$.

3. $y'' = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.

4. $y'' = \frac{ab}{(a^2 + x^2)^{\frac{3}{2}}}$.

5. $y'' = \frac{2}{x^3}$.

6. $y'' = \frac{2}{(x - 1)^3}$.

7. $y'' = -\frac{4}{(x + 1)^3}$.

8. $y'' = 6\left[\frac{3x^2 + 3x + 1}{(x^2 + x)^3}\right]$.

9. $-\frac{4p^2}{y^3}$.

10. $-\frac{r^2}{y^3}$.

11. $\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$.

12. $\frac{-b^4}{a^2y^3}$.

13. $\dfrac{-b^4}{a^2y^3}$.

14. $\dfrac{2(y - x - 4)}{(x - 4)^2}$.

15. $-\dfrac{1}{16}\left[\dfrac{4(y - 3)^2 + (1 - x)^2}{(y - 3)^3}\right]$.

16. $\dfrac{4y(y - 4x)}{(y - 2x)^3}$.

17. $y^{(n)} = (-1)^n \dfrac{n!}{x^{n+1}}$.

18. $y^{(n)} = (-1)^n \dfrac{(n + 1)!}{x^{n+2}}$.

20. $\dfrac{-48x}{y^5}$.

21. $-\dfrac{3b^6x}{a^4y^5}$.

22. -6.

Art. 36, p. 68

1. max. $(-1, 5)$
   min. $(1, 1)$
   inf. $(0, 3)$.

2. max. $(2, 0)$
   min. $(4, -4)$
   inf. $(3, -2)$.

3. min. $(0, -16)$.

4. max. at $x = 0$
   min at $x = \pm 4$.
   inf. at $x = \pm\dfrac{4}{\sqrt{3}}$.

5. max. $(-2, -2)$
   min. $(2, 2)$
   No inf. point.

6. max. $(-3, -36)$
   min. $(3, 36)$
   No inf. point.

7. No max. or min.
   inf. $(-3, -4)$.

8. max. $(-3, \frac{41}{3})$
   min. $(1, 3)$
   inf. $(-1, \frac{25}{3})$

9. max. $(0, 2a)$
   inf. $\left(\pm\dfrac{2a}{\sqrt{3}}, \frac{3}{2}a\right)$.

10. max. $(0, 4)$
   inf. $\left(\pm\dfrac{4}{\sqrt{3}}, 3\right)$

11. max. $(2, 4)$
   min. $(6, -4)$, $(-2, -4)$.
   inf. $x = 2 \pm \frac{4}{3}\sqrt{3}$.

12. min. $(2, -14)$
   inf. at $x = 0$ and $x = \frac{4}{3}$.

13. 6 m.p.h. per minute.

15. $v = 3\frac{1}{2}$ units per second.
   $a = 2\frac{1}{2}$ units per second per second.

CHAPTER VII

Art. 37, pp. 75–76

1. $\dfrac{2}{\sqrt{3}}$.

2. 0.

3. $\dfrac{2}{\sqrt{3}} - 1$.

11. $-\frac{120}{169}$, $\frac{119}{169}$, 5.

12. $\frac{63}{65}$, $-\frac{33}{56}$.

19. $\cos\theta = -1$, or $\frac{1}{2}$.

20. $\sec\theta = 2$, or $-\frac{5}{4}$.

21. $\sin x = \frac{1}{2}$, or $-\frac{1}{3}$.

24. $2^{\frac{3}{2}}\sqrt{1 - \cos\theta}$.

Art. 39, pp. 79–81

4. $12 \cos 3x$.

5. $2 - \sin x$.

6. $\cos\theta(\cos^2\theta - 2\sin^2\theta)$.

7. $-6 \cot x \csc^2 x$

8. $\frac{1}{2}\sin\theta$.

9. $\dfrac{\cos 2x}{\sqrt{\sin 2x}}$.

10. $\sin x(\tan^2 x + 2)$.

11. $\cos t - t \sin t$.

12. $2x \tan x(x \sec^2 x + \tan x)$.

13. $-\dfrac{4a \sin x}{(1 - \cos x)^2}$.

14. $\dfrac{2x \sec x \tan x + \sec x}{2\sqrt{x}}$.

15. $\dfrac{-x \sin x - \cos x}{x^2}$.

16. $\dfrac{1}{\cos x - 1}$.

17. $\dfrac{2 \sec x}{(\sec x - \tan x)^2}$.

18. $\frac{-2 \cos x}{(1 + \sin x)^2}$. **19.** $\frac{-1}{\sin^2 x \cos^2 x}$. **20.** $8x \sin (2x^2 + 12)$.

21. $\frac{4x^2 \sin (1 - x^2) + \cos (1 - x^2)}{2\sqrt{x}}$. **22.** $2 \cos 2x$.

23. $\frac{1}{4} \sec \frac{1}{2}x(\sec^2 \frac{1}{2}x + \tan^2 \frac{1}{2}x)$. **24.** $\cos x(9 \cos^2 x - 7)$.

25. $\frac{1}{2} \sec^2 \frac{1}{2}x(3 \sec^2 \frac{1}{2}x - 2)$. **26.** $\frac{\sin x}{(1 - \cos x)^2}$.

27. $\frac{2 \sin x - 2x \cos x - x^2 \sin x}{x^3}$. **30.** $\tan \varphi = \pm 2\sqrt{2}$.

31. $\tan \varphi = \pm \frac{3}{4}; \pm \frac{1}{3}$. **34.** max. at $x = \frac{\pi}{2}$.
min. at $x = 0$, $x = \pi$.
inf. at $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$.

35. max. at $x = \frac{\pi}{2}$.
min. at $x = 0, \pi$.
inf. at $x = \frac{\pi}{3}$.

36. max. at $x = \frac{\pi}{4}$.
min. at $x = \frac{5\pi}{4}$.
inf. at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

38. min. at $x = \frac{\pi}{6}$.
max. at $x = -\frac{\pi}{6}$.
inf. at $x = 0$.

40. $\frac{3}{25}$ radian per second. **41.** $\frac{\pi}{2}$.

42. $\tan \theta = 3$; $L = 10\sqrt{10}$ ft. **43.** No. Longest is $\frac{13\sqrt{13}}{2} = 23.4$ ft.

44. $\frac{\pi}{3}$. **45.** $\frac{4}{3}a$. **46.** $\frac{1}{3}a$.

48. $\tan \theta = \sqrt{2}$.

Art. 41, pp. 86–87

5. $-\frac{4}{\sqrt{1 - 16x^2}}$. **6.** $-\frac{1}{\sqrt{4 - x^2}}$. **7.** $\frac{1}{2\sqrt{x - x^2}}$.

8. $\frac{1}{x\sqrt{x^2 - 1}}$. **9.** $\frac{1}{1 + x^2}$. **10.** $\frac{8x}{\sqrt{1 - x^4}}$.

11. $\frac{3}{1 + 9x^2}$. **12.** $-\frac{2}{4 + \theta^2}$. **13.** $\frac{1}{\sqrt{1 - x^2}}$.

14. $\frac{1}{1 + x^2}$. **15.** $-\frac{1}{(1 + x)\sqrt{x}}$.

16. $\frac{x^2}{\sqrt{1 - x^2}} + 2x \operatorname{arc} \sin x$. **17.** $\frac{2x^2}{1 + x^4} + \operatorname{arc} \tan x^2$.

18. $-\frac{x}{\sqrt{1 - x^2}} + \operatorname{arc} \cos x$. **19.** $x(4 - x^2)^{-\frac{3}{2}}$.

20. $\frac{-108x.}{(1 - 9x^2)^{\frac{3}{2}}}$. **21.** $\frac{-16x}{(1 + 4x^2)^2}$. **22.** $\frac{4x}{(x^2 + 4)^2}$

23. $\frac{2x}{(1 + x^2)^2}$. **24.** $-\frac{x}{(1 - x^2)^{\frac{3}{2}}}$. **25.** $(0, 0)$.

28. $-\frac{1}{20}$ radian per minute. **29.** $-\frac{3}{50}$ radian per second.

30. $4y - 2x = \pi - 2$. **31.** $\tan \varphi = \pm\frac{1}{3}$. **32.** 12 ft.
33. 9 ft. per second.

CHAPTER VIII

Art. 45, pp. 92–93

2. $\frac{1}{2}(e^x + e^{-x})$. **4.** 1.593.

Art. 47, pp. 95–96

4. -0.693 unit per minute. **5.** $\frac{3}{x}$.

6. $\frac{3}{6x + 8}$. **7.** $\frac{-16x}{(x^2 + 4)(x^2 - 4)}$. **8.** $\cot x$.

9. $\frac{2 \log_{10} e \log_{10} x}{x}$. **10.** $\frac{-x \log_a e}{r^2 - x^2}$. **11.** $2 \tan x$.

12. $\frac{1}{\sqrt{x^2 - a^2}}$. **13.** $\sec x$. **14.** $\frac{1}{x \log x}$. **15.** $\frac{2x^2 - 4}{x(x^2 - 4)}$.

16. $\frac{1}{1 - t^2}$. **17.** $\frac{2x}{1 - x^4}$. **18.** $\frac{2}{x} + \cot x$. **19.** $\frac{5x - 1}{2(x^2 - 1)}$.

21. $2e^{\frac{1}{2}x}$. **22.** $-2xe^{-x^2}$. **23.** $(3 \log 2)2^{3x}$.

24. $10^{x^2-1} (\log 10)\ (2x)$. **25.** $\frac{2x}{9}3^{x^2} \log 3$. **26.** $-(2e)^{-x} \log (2e)$.

27. $e^{-x}(1 - x)$. **28.** $8xe^{x^2}(x^2 + 1)$. **29.** $e^x\left(\frac{1}{x} + \log x\right)$.

30. $2e^{-x}(\cos x - \sin x)$. **31.** $\frac{1}{2}(e^x - e^{-x})$.

32. $\frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$. **33.** $-\frac{1}{2}e^{-\frac{1}{2}x}\left(\pi \sin \frac{\pi x}{2} + \cos \frac{\pi x}{2}\right)$.

37. min. (e, e).
inf. $(e^2, \frac{1}{2}e^2)$.
38. min. $\left(\frac{1}{e}, -\frac{1}{e}\right)$.
39. max. at $x = 1$.
inf. at $x = 2$.

40. max. at $x = 0$
inf. at $x = \pm\frac{1}{\sqrt{2}}$.
41. $\tan \varphi = \pm\frac{1}{3}$.

Art. 50, pp. 98–99

1. $\frac{y(2x^2 - 1)}{x(x^2 - 1)}$. **2.** $y\left[\frac{1}{t} + \frac{1}{2(t + 1)} + \frac{1}{3(t - 1)}\right]$.

3. $\frac{y}{2}\left(\frac{1}{x + 2} + \frac{1}{x + 3} - \frac{1}{x} - \frac{1}{x + 1}\right)$. **4.** $\frac{2xy}{1 - x^4}$.

5. $y\left(\frac{2a^2x}{a^4 - x^4} - \frac{1}{x}\right)$. **6.** $2y\left[\frac{3x}{3x^2 + 4} - \frac{x}{3(x^2 - 6)}\right]$.

7. $y(1 + \log x)$. **8.** $\frac{y}{x^2}(1 - \log x)$. **9.** $\frac{2y \log x}{x}$.

10. $y\left(\frac{\sin x}{x} + \cos x \log x\right)$. **11.** $y\left(\frac{1}{\log x} + \log \log x\right)$.

12. $y\ (\log 2 - \log x - 1)$.

CHAPTER IX

Art. 52, pp. 102–103

4. $\sqrt{1 + \cos^2 x}$. **5.** $\frac{1}{2}\sqrt{\frac{4x + 1}{x}}$. **6.** $\frac{\sqrt{x^2 + 1}}{x}$. **7.** $\frac{r}{y}$.

8. $\sqrt{\frac{2 - x^2}{1 - x^2}}$. **9.** $\sec x$.

10. $3\frac{1}{2}$. **11.** $3\sqrt{2}$ units per second.

Art. 56, pp. 108–109

1. 0.573 degree per foot. **2.** 11.5 degrees per 100 feet.

3. $R = \frac{1}{2}(5)^{\frac{3}{2}}$. **4.** $R = \frac{1}{2}$. **5.** $R = \frac{10^{\frac{3}{2}}}{8}$. **6.** $R = -2^{\frac{3}{2}}$.

7. $R = -\frac{b}{a^2}; \frac{b^2}{a}$. **8.** $R = -\frac{4(10)^{\frac{3}{2}}}{5}$. **9.** $R = -\frac{(e^2 + 1)^{\frac{3}{2}}}{e}$.

10. $R = -1$. **11.** $R = -\frac{7^{\frac{3}{2}}}{4}$. **12.** $R = -\sec x_1$. **13.** $R = \sec x_1$.

14. $R = -\frac{1}{2}$. **15.** $K = \frac{2x}{(1 + x^4)^{\frac{3}{2}}}$.

16. $x = \frac{\sqrt{2}}{2}$; $|R| = \frac{3}{2}\sqrt{3}$. **17.** $\frac{1}{2}$.

18. a. **19.** -12.

21. $\begin{cases} \alpha = x - \dfrac{y'(1 + y'^2)}{y''}. \\ \beta = y + \dfrac{(1 + y'^2)}{y''}. \end{cases}$ **22.** $(3, -2)$.

CHAPTER X

Art. 58, p. 112

1. $y'' = -\frac{1}{a}\csc^3 \theta$. **2.** $y'' = -\frac{3 \csc^3 \theta}{16}$. **3.** $y'' = -\frac{1}{8}$.

4. $y'' = -\frac{\cos \theta\,(1 + 2 \sin^2 \theta)}{4 \sin^3 \theta}$. **5.** $y'' = -4 \cos^3 \theta \sin 3\theta$.

6. $y'' = \frac{1}{12a \cos^4 \theta \sin \theta}$. **7.** $8x - y = 12$.
$x + 8y = 99$.
$y = x^2 + 2x - 3$.

8. $(-1, -4)$; $R = \frac{1}{2}$. **11.** $-2p \sec^3 \theta$. **12.** $-\frac{b}{a^2}\cot^3 \theta$; $-\frac{b^4}{a^2 y^3}$.

13. max. $(0, 4)$
inf. $\left(\pm\frac{4}{\sqrt{3}}, 3\right)$. **14.** $-4a$.

Art. 59, pp. 116–117

2. $v = \sqrt{117}$; $a = \sqrt{445}$. **3.** $v = 8$; $a = \sqrt{1040}$.
$4y = x^2 - 8x - 48$.

4. $v = \sqrt{13}$; $a = 6$. $8y = (x - 1)^3$.
5. $v = 4$; $a = 4$. $x^2 + y^2 = 16$.
6. $x^2 + y^2 = 4y$.
7. $v = 4$; $a = 8$. $x^2 + 4y^2 = 16x$.
12. $v_x = v_0 \cos \alpha$; $v_y = v_0 \sin \alpha - gt$. $a_x = 0$; $a_y = -g$.
13. $\dfrac{v_0^2 \sin^2 \alpha}{2g}$.
14. $\dfrac{\pi}{4}$.

CHAPTER XI

Art. 60, pp. 120–123

16. $12x^2 - 7$.
17. $12x(3x^2 + 5)$.
18. $\dfrac{-2}{(3 - x)^2}$.
19. $\dfrac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$.
20. $\dfrac{6x(x + 1)}{(2x + 1)^2}$.
21. $\dfrac{-3y}{(3x + 1)(3x - 1)}$.
22. $\dfrac{(3x + 5)}{x^2 + 2x}$.
23. $x(1 + 2 \log x)$.
24. $\dfrac{2 \log (3x)}{x}$.
25. $e^{-x^2}(1 - 2x^2)$.
26. $10^{x+4} \log 10$.
27. $y\left(\dfrac{1}{\log x} + \log \log x\right)$.
28. $y\left[\dfrac{x}{x + 4} + \log (x + 4)\right]$.
29. $4 \sin 2x \cos 2x$.
30. $-\frac{3}{2} \sin x$.
31. $\cos x (\cos^2 x - 2 \sin^2 x)$.
32. $\sec^3 x$.
33. $\dfrac{1}{2\sqrt{x - x^2}}$.
34. $-\dfrac{20}{x^2 + 400}$.
35. $\dfrac{-r}{r^2 + (r - x)^2}$.
36. $-\dfrac{9x}{16y}$.
37. $-\dfrac{2xy}{x^2 + 1}$.
38. $-\sqrt{\dfrac{y}{x}}$.
39. $\dfrac{y - x^2}{y^2 - x}$.
40. $\dfrac{\cos x - y}{x}$.
41. $\dfrac{1 - xy}{x^2}$.
42. $\dfrac{4 - y^2 - 8xy}{3y^2 + 4x^2 + 2xy}$.
43. $3 + 2 \log x$.
44. $\dfrac{2 - x^2}{(1 - x^2)^{\frac{3}{2}}}$.
45. $-\dfrac{4}{y^3}$.
46. $-\dfrac{2xy^3 + 2(1 - x^2)^2}{y^5}$.
47. $2e^{-x^2}(2x^2 - 1)$.
48. $2 \sec^2 x(3 \sec^2 x - 2)$.
49. $-\dfrac{3 \csc^3 \theta}{16}$; $-\dfrac{81}{16y^3}$.
50. $2 \sin^4 \theta(3 \cos^2 \theta - \sin^2 \theta)$; $\dfrac{2(3x^2 - 1)}{(1 + x^2)^3}$.
52. $y - 3x + 15 = 0$.
53. $\tan \varphi = \pm\frac{3}{4}$; $\pm\frac{3}{2}$.
54. $\tan \varphi = \pm\frac{4}{3}$.
55. $-\dfrac{(73)^{\frac{3}{2}}}{100}$.
56. $x = \frac{3}{2}$.
57. max. $(2, 1)$. min. $(-2, -1)$. inf. $(0, 0)$; $(\pm 2\sqrt{3}, \pm\frac{1}{2}\sqrt{3})$.
58. min. $\left(\dfrac{1}{e}, -\dfrac{1}{e}\right)$.
59. $\frac{1}{2}$.
60. 2:3.
62. $AP = AQ = 6$ in.
63. length $= 33\frac{1}{3}$ in.
64. $OB = x = 3 \cos \theta + \sqrt{25 - 9 \sin^2 \theta}$.

$$\frac{dx}{dt} = -6\pi\left(\sin \theta + \frac{3 \sin \theta \cos \theta}{\sqrt{25 - 9 \sin^2 \theta}}\right).$$

65. 4.7 ft. per second.

66. $\cos^3\theta = \frac{a}{L}$ where θ is inclination of line.

68. Width at top = twice slant height of bank.

CHAPTER XII

Art. 62, pp. 128–129

4. $\frac{3}{2}$. **5.** 4. **10.** 2. **11.** $\frac{1}{2}$.
12. $\frac{1}{2}$. **13.** $\frac{1}{6}$. **14.** 1. **15.** 0.
16. $\frac{1}{2}$. **17.** 2. **18.** log 3. **19.** $-\frac{1}{2}$.
20. 2. **21.** 2. **22.** $\frac{1}{2}$. **23.** No limit.
24. 0. **25.** $\frac{1}{2}$. **26.** No limit. **27.** 0.
28. No limit. **31.** 2π. **32.** 0. **33.** $\frac{1}{4}$.
34. $-\frac{2}{\pi}$.

CHAPTER XIII

Art. 68, pp. 137–138

3. 400 sq. yd. **4.** πDLt. **5.** 4π sq. in. **6.** $6x^2t$.
7. 1.035. **8.** 0.47 **9.** 0.0098. **10.** 2.0086.
11. $3\frac{1}{27}$. **12.** 2.025. **13.** 19.75. **14.** 77.76.
15. 5.067. **16.** 67.84. **17.** 2.993. **18.** 9.49.
19. 0.4697 **20.** 0.965. **21.** $\frac{\sin\theta}{1-\cos\theta}$. **22.** $\frac{\cos t - \cos 2t}{\sin 2t - \sin t}$.
23. $\frac{r}{2}$. **24.** $\frac{x}{4}$.

Art. 69, pp. 139–140

1. 40 sq. ft. **2.** 13 sq. ft. **3.** 6 sq. in. **4.** 0.033 in.
5. $r = 2.224 \pm 0.004$ in. **6.** $r <$ about 9 in.
7. $\theta <$ about 78°. **8.** $-\sin\theta\, d\theta$.
11. 3 per cent; 2 per cent. **13.** 0.28 per cent.

CHAPTER XIV

Art. 74, pp. 147–149

1. $\frac{3x^5}{5} + \frac{4x^3}{3} + 2x + C$. **2.** $\frac{1}{3}(2x)^{\frac{3}{2}} + C$.
3. $\frac{3}{16}(4x+1)^{\frac{4}{3}} + C$. **4.** $-\frac{1}{2x^2} + C$. **5.** $18t^{\frac{1}{3}} + C$.
6. $\frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + C$. **7.** $\frac{4}{5}s^{\frac{5}{4}} + C$. **8.** $-\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}} + C$.
9. $\frac{x^5}{5} + \frac{x^4}{2} + 3x^3 + 4x^2 + 16x + C$. **10.** $\frac{1}{2}\sin^2 x + C$.
11. $\frac{1}{9}\tan^3 3x + C$. **12.** $\frac{\log^3 x}{3} + C$. **13.** $-\frac{2}{x-3} + C$.
14. $ax - \frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{1}{2}x^2 + C$. **15.** $2(e^x + 4)^{\frac{1}{2}} + C$.

16. $\frac{3}{2}(\text{arc tan } x)^2 + C.$
17. $\frac{x^5}{5} + 2x^3 + 12x - \frac{8}{x} + C.$
18. $\frac{4}{7}x^{\frac{7}{4}} + \frac{4}{5}x^{\frac{5}{4}} + \frac{2}{3}x^{\frac{3}{2}} + C$
19. $12x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + C.$
20. $\frac{2}{3}x^{\frac{3}{2}} - \frac{8}{3}x^{\frac{3}{4}} - \frac{1}{x} + C.$
21. $\frac{1}{2}\log(2x + 1) + C.$
22. $\frac{1}{6}\log(3x^2 - 1) + C.$
23. $\frac{1}{3}\log(\tan\theta + 6) + C.$
24. $-\log\cos x + C.$
25. $\log\sin x + C.$
26. $\frac{1}{2}\log(x^2 - 6x + 4) + C.$
27. $-\frac{1}{2}\log(4\cos\frac{1}{2}x + 3) + C.$
28. $\frac{\log(2^x + 3)}{\log 2} + C.$
29. $\frac{1}{3}\log(3e^x + 4) + C.$
30. $\log\log s + C.$
31. $\frac{3}{4(4 - 2x^2)} + C.$
32. $\frac{x^7}{7} + \frac{4x^5}{5} + \frac{4x^3}{3} + C.$
33. $-(a^2 - x^2)^{\frac{1}{2}} + C.$
34. $\frac{(1 + \tan\theta)^{-2}}{-2} + C.$
35. $-\frac{1}{8}\log(3 + 4\cot 2x) + C.$
36. $\frac{2}{3}\sin^3 x + C.$
37. $\frac{1}{2}\log^2\sin\theta + C.$
38. $\frac{1}{2}(\log\tan\frac{1}{2}\theta)^2 + C.$
39. $\log(\sec\theta + \tan\theta) + C.$
40. $\log(\csc\theta - \cot\theta) + C.$

Art. 75, pp. 149–150

1. $\frac{1}{2}e^{2x} + C.$
2. $\frac{4^{3x+5}}{3\log 4} + C.$
3. $3e^{\tan 2\theta} + C.$
4. $\frac{1}{2}e^{x^2} + C.$
5. $\frac{(ae)^x}{1 + \log a} + C.$
6. $-\frac{4e^{-x^3}}{3} + C.$
7. $2(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}) + C.$
8. $\frac{1}{2}(e^{2x} - e^{-2x}) + 2x + C.$
9. $\frac{a^2}{4}(e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}) + ax + C.$
10. $\frac{1}{2}x^2 + C.$
11. $-\frac{10^{\frac{1}{x^2}}}{2\log 10} + C.$
12. $e^{\sin x\cos x} + C.$
13. $\frac{1}{2}e^{1+\sin 2x} + C.$
14. $\frac{2^{\text{arc tan } x}}{\log 2} + C.$
15. $-\frac{1}{3}e^{-x^3} - e^{-x} + C.$
17. $\frac{1}{2}\sin 2x + C.$
18. $\frac{1}{4}\tan(4x + 2) + C.$
19. $-\frac{1}{2}\cot(x^2 + 1) + C.$
20. $2\sin\frac{1}{2}x + C.$
21. $\frac{1}{3}\sec 3\theta + C.$
22. $\frac{1}{6}\tan 2t + C.$
23. $2\sin\frac{1}{2}x + C.$
24. $-\frac{3}{\pi}\cos\pi x + C.$
25. $x + \sin^2 x + C.$
26. $\frac{1}{3}\log[\sec(3x + 4) + \tan(3x + 4)] + C.$
27. $\frac{1}{2}\log(\csc x^2 - \cot x^2) + C.$
28. $\frac{1}{3}\log(\csc 3x - \cot 3x) + C.$
29. $\frac{1}{\pi}\log(\sec\pi\theta + \tan\pi\theta) + C.$
30. $\frac{1}{2}\tan\theta + C.$
31. $2\sin x - \log(\sec x + \tan x) + C.$
32. $2\log(\csc\frac{1}{2}x - \cot\frac{1}{2}x) + 4\cos\frac{1}{2}x + C.$
33. $2\log(\sec\theta + \tan\theta) + C.$
34. $\sec x + C.$
35. $-8\cos x + C.$
36. $-2\csc x + C.$
37. $\frac{1}{2\pi}\sin^2\pi x + C.$
39. $-\frac{2\cos^3\theta}{3} + \cos\theta + C.$
40. $-\log(2 - \sec^2\theta) + C.$

Art. 76, pp. 151–152

1. $\frac{1}{6}\text{ arc tan }\frac{3x}{2}+C.$

2. $\text{arc sin }\frac{x}{3}+C.$

3. $\frac{1}{2}\text{ arc sin }\frac{2x}{\sqrt{10}}+C.$

4. $\frac{1}{16}\log\frac{x^2-4}{x^2+4}+C.$

5. $-\log(\cos x+\sqrt{\cos^2 x+16})+C.$

6. $\frac{1}{2}\text{ arc tan }\frac{e^x}{2}+C.$

7. $\frac{1}{b}\text{ arc sin }\frac{bx}{a}+C.$

8. $\frac{1}{3}\text{ arc tan }\frac{\log x}{3}+C.$

9. $\frac{1}{12}\log\frac{t^2+1}{t^2+7}+C.$

10. $\frac{1}{2}\text{ arc sin }\frac{2x+1}{\sqrt{10}}+C.$

11. $\frac{1}{2}\text{ arc tan }\frac{x+1}{2}+C.$

12. $\frac{1}{2\sqrt{6}}\text{ arc tan }\frac{2t-\text{[illegible]}}{\sqrt{6}}+C.$

13. $-\text{ arc sin }(1-x)+C.$

14. $\text{arc sin }\frac{s-3}{2}+C.$

15. $\frac{1}{2\sqrt{10}}\text{ arc tan }\frac{x^2+2}{\sqrt{10}}+C.$

16. $\frac{1}{4}\log\frac{2t-3}{2t+1}+C.$

17. $\frac{1}{2}\text{ arc sin }\frac{2t^2-3}{4\sqrt{2}}+C.$

18. $\log(z+1+\sqrt{z^2+2z+7})+C.$

19. $\frac{1}{\log 2}\text{ arc sin }\frac{2^x}{3}+C.$

20. $\frac{1}{4}\log(4x-3+\sqrt{16x^2-24x-7})+C.$

21. $u+C.$

22. $\log\frac{e^{\frac{x}{2}}-1}{e^{\frac{x}{2}}+1}+C.$

23. $\log(x^2+4x+8)+\text{arc tan }\frac{x+2}{2}+C.$

24. $\log(4x^2-4x-3)+\frac{3}{8}\log\frac{2x-3}{2x+1}+C.$

25. $2(9x^2-12x)^{\frac{1}{2}}+3\log(3x-2+\sqrt{9x^2-12x})+C.$

27. $\frac{1}{2}\log(x^2+4x-5)+\frac{1}{6}\log\frac{x-1}{x+5}+C.$

28. $\frac{3}{2}\log(x^2+x+1)+\frac{7}{\sqrt{3}}\text{ arc tan }\frac{2x+1}{\sqrt{3}}+C.$

29. $-\frac{1}{9}(3-6x-9x^2)^{\frac{1}{2}}-\frac{22}{9}\text{ arc sin }\frac{3x+1}{2}+C.$

30. $(4x^2-12x+5)^{\frac{1}{2}}+\frac{7}{2}\log(2x-3+\sqrt{4x^2-12x+5})+C.$

31. $4(x^2-3x+2)^{\frac{1}{2}}+13\log(x-\frac{3}{2}+\sqrt{x^2-3x+2})+C.$

32. $(x^2-5x+1)^{\frac{1}{2}}+\frac{17}{2}\log(x-\frac{5}{2}+\sqrt{x^2-5x+1})+C.$

33. $\frac{1}{18}\left[-7\log(9x^2-6x-15)-\frac{13}{4}\log\frac{3x-5}{3x+3}\right]+C.$

Art. 77, pp. 152–153

1. $-\csc x+C.$

2. $\sec x+C.$

3. $\tan\theta-2\log\cos\theta+C.$

4. $\frac{x^2}{2}+6x+12\log x-\frac{8}{x}+C.$

5. $\dfrac{x^2}{2} - \frac{4}{3}x^{\frac{3}{2}} + \dfrac{x^3}{3} + C.$

6. $\dfrac{\sin^4 x}{4} + C.$

7. $\dfrac{\cos^3 x}{3} - \cos x + C.$

8. $\dfrac{x^4}{4} - \cos x + C.$

9. $\sin^2 x - \dfrac{\cos^4 x}{4} + C.$

10. $\frac{1}{6}(1 + \sin^2 x)^3 + C.$

11. $\dfrac{\sec^2 x}{2} + C.$

12. $\tan x - \cot x + C.$

13. $\log (4 - \cos^2 x) + C.$

14. $\frac{1}{4} \log \dfrac{\cos x - 2}{\cos x + 2} + C.$

15. $\log (\sin x + \cos x) + C.$

16. $\dfrac{(4 + 2^x)^2}{2 \log 2} + C.$

17. $e^{\tan x} + C.$

18. $2x + \frac{1}{2}(e^{2x} - e^{-2x}) + C.$

19. $x^2 + 3 \log (x + 2) + C.$

20. $2x + 3 \log (x + 4) + C.$

21. $\dfrac{x^3}{3} + \dfrac{3x^2}{2} + x + 2 \log (x + 1) + C.$

22. $\dfrac{x^2}{2} + 2x + \frac{1}{2} \arctan \dfrac{x}{2} + C.$

23. $\frac{1}{2} \log (x^2 + 4x + 8) + \arctan \dfrac{x + 2}{2} + C.$

24. $-\frac{1}{4}(x^2 + 4x + 8)^{-2} + C.$

25. $\frac{1}{4}(6 + \log^2 x)^2 + C.$

26. $\frac{1}{3} \arctan 3x + C.$

27. $-\frac{1}{2}(1 - 4x^2)^{\frac{1}{2}} + 2 \arcsin 2x + C.$

28. $-e^{-\sin x} + C.$

29. $2e^{\sqrt{x}} + C.$

30. $8 \log (3 + x^{\frac{1}{4}}) + C.$

31. $2 \arctan \dfrac{\tan x}{2} + C.$

32. $\frac{1}{2} \arctan \log t + C.$

33. $\arcsin \dfrac{e^{\sin x}}{2} + C.$

34. $\arcsin (\frac{1}{2} \log x) + C.$

35. $e^x - 4 \log (e^x + 4) + C.$

36. $\frac{2}{3}(\log x - 7)^{\frac{3}{2}} + C.$

37. $\dfrac{-4}{\sqrt{3}}(1 - \sqrt{x})^{\frac{3}{2}} + C.$

38. $\dfrac{(2\pi)^{5x}}{5 \log 2\pi} + C.$

39. $\frac{1}{2} \tan 2x - x + C.$

40. $\csc x - \cot x + C.$

41. $\frac{1}{8} \log (4x^2 - 4x + 7) - \dfrac{5}{4\sqrt{6}} \arctan \dfrac{2x - 1}{\sqrt{6}} + C.$

42. $3(x^2 + x + 1)^{\frac{1}{2}} - \frac{5}{2} \log (x + \frac{1}{2} + \sqrt{x^2 + x + 1}) + C.$

43. $-\frac{1}{4}(12x - 4x^2)^{\frac{1}{2}} + \frac{11}{4} \arcsin \dfrac{2x - 3}{3} + C.$

44. $\frac{3}{2}(4x^2 - 4x + 10)^{\frac{1}{2}} + \log (2x - 1 + \sqrt{4x^2 - 4x + 10}) + C.$

CHAPTER XV

Art. 79, p. 156

1. $3y = x^3 + 12x - 36.$

3. $y = e^x.$

4. $y = \dfrac{1}{2 - 3x}.$

5. $x^2 + y^2 = 25.$

6. $y = x^3 - 3x^2 - 4x + 12.$

7. $y = 4 - \frac{1}{2} \log x.$

8. $x^2 - y^2 = 12.$ 9. $y - 1 = C(x - 1).$ 10. $xy = C.$
11. $y + \log(y - 1) = \frac{1}{2}x^2.$ 12. $3y = 2x^3 - 3x + 13.$

Art. 80, pp. 159–160

1. $\frac{1}{2}$ g. 2. 3.32 hr. 3. 17.3 years. 4. \$546.63, \$542.62
5. $8\frac{1}{2}$ years. 6. 9.47 gal., 7.89 gal. 7. 149 min. 8. $141\frac{1}{4}$ min.
9. 204.3 min. 10. 185.4 min. 11. 8 min. 12. 45°
13. 21.7 hr.

Art. 81, pp. 163–164

1. $v = 80 - gt;$
$y = 50 + 80t - \frac{1}{2}gt^2.$
149.4 ft.

2. $v = -60 - gt.$
$y = 800 - 60t - \frac{1}{2}gt^2.$

4. $v_x = 100$
$v_y = \sqrt{12{,}000g}.$

6. $\dfrac{v - 10}{v + 10} = \dfrac{1}{3}e^{-\frac{gt}{5}}.$

9. $v = \dfrac{w}{k}\left(1 - e^{-\frac{gkt}{w}}\right)$

10. $v_x = \dfrac{v_0}{1 + v_0k't};\ v_y = -gt.$

$x = \dfrac{1}{k'}\log(1 + v_0k't);\ y = -\frac{1}{2}gt^2.$

$\left(k' = \dfrac{kg}{w}\right).$

11. $v = 5\left(1 - e^{-\frac{gt}{100}}\right)$

$x = 5t + \dfrac{500}{g}\left(e^{-\frac{gt}{100}} - 1\right).$

CHAPTER XVI

Art. 83, p. 167

1. $\frac{1}{3}\cos^3 x - \cos x + C.$
2. $2\sin\frac{1}{2}x - \frac{2}{3}\sin^3\frac{1}{2}x + C.$
3. $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$
4. $\frac{1}{2}\sin 2x - \frac{1}{3}\sin^3 2x + \frac{1}{10}\sin^5 2x + C.$
5. $-\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.$
6. $\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.$
7. $-\frac{1}{6}\cos^6 x + C.$
8. $\frac{1}{8}\sin^4 2\theta - \frac{1}{12}\sin^6 2\theta + C.$
9. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$
10. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$
11. $\frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C.$
12. $\frac{2}{3}(\sin x)^{\frac{3}{2}} - \frac{2}{7}(\sin x)^{\frac{7}{2}} + C.$
13. $\frac{1}{2}\sec^2 x + \log\cos x + C.$
14. $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.$
15. $\sec x + C.$
16. $\frac{1}{3}\sec^3\theta + C.$
17. $\frac{1}{3}\sec^3 x - \sec x + C.$
18. $-2\sqrt{\cos x} + \frac{2}{5}(\cos x)^{\frac{5}{2}} + C.$
19. $\frac{1}{2}x - \frac{1}{4}\sin 2x + C.$
20. $\frac{1}{2}(\theta + \sin\theta) + C.$
21. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$
22. $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$
23. $\frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$
24. $\frac{5}{16}x + \frac{1}{2}\sin x + \frac{3}{32}\sin 2x - \frac{1}{24}\sin^3 x + C.$

25. $\frac{1}{8}\theta - \frac{1}{16}\sin 2\theta + C.$
26. $\frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$
27. $\frac{1}{128}(3x - \sin 4x + \frac{1}{8}\sin 8x) + C.$
28. $\frac{1}{16}(x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x) + C.$
29. $x + \sin^2 x + C.$
30. $\frac{3}{2}x - \cos 2x - \frac{1}{8}\sin 4x + C.$
31. $x + \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + 2\sin x - \frac{4}{3}\sin^3 x + C.$
32. $2x + \sin 2x - \cos x + \frac{8}{3}(\sin x)^{\frac{3}{2}} + C.$
33. $\frac{2}{3}\sin^3 x + C.$
34. $\frac{1}{4}\sin^4 x + C.$
35. $-\frac{1}{20}\cos^5 2x + C.$
36. $\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{2}\sin\theta + \frac{1}{6}\sin^3\theta + C.$
37. $\frac{1}{8}\theta - \frac{1}{32}\sin 4\theta + C.$

Art. 86, pp. 169–170

1. $\tan x - x + C.$
2. $\frac{1}{2}\tan^2 x + \log\cos x + C.$
3. $-2\cot\frac{1}{2}\theta - \theta + C.$
4. $-\frac{1}{4}\cot^2 2\theta - \frac{1}{2}\log\sin 2\theta + C.$
5. $\frac{1}{3}\tan^3\theta - \tan\theta + \theta + C.$
6. $-\frac{2}{3}\cot^3\frac{1}{2}x + 2\cot\frac{1}{2}x + x + C.$
7. $\frac{1}{8}\tan^4 2x - \frac{1}{4}\tan^2 2x - \frac{1}{2}\log\cos 2x + C.$
8. $2\tan\frac{1}{2}x + C.$
9. $\frac{2}{3}\tan^3\frac{1}{2}x + 2\tan\frac{1}{2}x + C.$
10. $\frac{1}{4}\tan 4x + \frac{1}{6}\tan^3 4x + \frac{1}{20}\tan^5 4x + C.$
11. $-\cot x + C.$
12. $\tan x + \frac{1}{3}\tan^3 x + C.$
13. $\frac{1}{4}\tan^4 x + C.$
14. $-\frac{1}{4}\cot^4 x + C.$
15. $\frac{1}{6}\tan^6 x + \frac{1}{2}\tan^4 x + \frac{1}{2}\tan^2 x + C.$
16. $\frac{1}{14}\tan^7 2x + \frac{1}{10}\tan^5 2x + C.$
17. $\frac{1}{5}\tan^5 x + C.$
18. $\frac{1}{14}\tan^7 2x + \frac{1}{5}\tan^5 2x + \frac{1}{6}\tan^3 2x + C.$
19. $\frac{1}{3}\sec^3 x - \sec x + C.$
20. $\frac{1}{3}\csc^3 x - \frac{1}{5}\csc^5 x + C.$
21. $\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.$
22. $\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C.$
23. $\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C.$
24. $\frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 x + C.$
25. $\frac{1}{2}\sec^4 x + C.$
26. $\frac{1}{3}\tan^3 x + C.$
27. $\frac{2}{3}\cos^3 x - 2\cos x + C.$
28. $\frac{2}{7}(\tan x)^{\frac{7}{2}} + \frac{2}{3}(\tan x)^{\frac{3}{2}} + C.$
29. $\tan x - \cot x + C.$
30. $13\tan x + 12\sec x - 9x + C.$

CHAPTER XVII

Art. 88, pp. 172–173

1. $\log(v + \sqrt{v^2 - a^2}) + C.$
2. $\frac{v}{2}\sqrt{a^2 - v^2} + \frac{a^2}{2}\arcsin\frac{v}{a} + C.$
3. $\frac{\sqrt{x^2 + 4}(x^2 - 8)}{3} + C.$
4. $-\frac{\sqrt{a^2 + x^2}}{a^2 x} + C.$
5. $\frac{\sqrt{x^2 + 5}(2x^2 - 5)}{75x^3} + C.$
6. $-\frac{(x^2 + 9)^{\frac{3}{2}}}{27x^3} + C.$
7. $\frac{3x^2 - 32}{15}(16 + x^2)^{\frac{3}{2}} + C.$
8. $\frac{9}{8}\arcsin\frac{x}{\sqrt{3}} + \frac{x\sqrt{3 - x^2}(2x^2 - 3)}{8} + C.$
9. $-\frac{1}{5}(9 - x^2)^{\frac{3}{2}}(x^2 + 6) + C.$
10. $-\frac{\sqrt{4 - 9x^2}}{x} - 3\arcsin\frac{3x}{2} + C.$
11. $\arcsin\frac{x}{a} + C.$
12. $-\frac{\sqrt{a^2 - x^2}(x^2 + 2a^2)}{3} + C.$
13. $\frac{1}{2}\log\frac{2 - \sqrt{4 - t^2}}{t} + C.$
14. $\sqrt{x^2 - 4} - 2\arccos\frac{2}{x} + C.$

15. $\frac{1}{5}(z^2 - 6)^{\frac{3}{2}}(z^2 + 4) + C.$

16. $\frac{(y^2 - 16)^{\frac{3}{2}}}{48y^3} + C.$

17. $\frac{1}{3}\sqrt{x^2 - 4}(x^2 + 8) + C.$

18. $\frac{1}{a} \text{ arc cos } \frac{a}{u} + C.$

19. $\frac{\sqrt{t^2 - 16}}{16t} + C.$

20. $\frac{\sqrt{x^2 - 8}}{16x^2} + \frac{1}{32\sqrt{2}} \text{ arc cos } \frac{2\sqrt{2}}{x} + C.$

21. $-\frac{1}{3}(4 - x^2)^{\frac{3}{2}} + C.$

22. $\frac{1}{3}(x^2 - 16)^{\frac{3}{2}} + C.$

23. $-\frac{1}{15}(a^2 - x^2)^{\frac{3}{2}}(5x^2 + 2a^2) + C.$

24. $\log (x + 1 + \sqrt{x^2 + 2x + 5}) + C.$

25. $\frac{(2x^2 + x - 9)\sqrt{3 - 2x - x^2}}{6} - 2 \text{ arc sin } \frac{x + 1}{2} + C.$

26. $\frac{x}{\sqrt{a^2 - x^2}} - \text{arc sin } \frac{x}{a} + C.$

27. $\frac{-x}{\sqrt{x^2 + a^2}} + \log (x + \sqrt{x^2 + a^2}) + C.$

28. $\frac{x^3}{3a^2(a^2 - x^2)^{\frac{3}{2}}} + C.$

Art. 90, pp. 174–175

1. $x - 8\sqrt{x} + 32 \log (\sqrt{x} + 4) + C.$
2. $x + 6\sqrt{x} + 6 \log (\sqrt{x} - 1) + C.$
3. $2\left(\sqrt{x + 1} + \log \frac{\sqrt{x + 1} - 2}{\sqrt{x + 1} + 2}\right) + C.$
4. $2\sqrt{x + 4} - 16 \log (\sqrt{x + 4} + 8) + C.$
5. $\frac{1}{7}(x - 1)(3x + 4)^{\frac{4}{3}} + C.$
6. $2t^{\frac{1}{2}} - 3t^{\frac{1}{3}} + 6t^{\frac{1}{6}} - 6 \log (t^{\frac{1}{6}} + 1) + C.$
7. $2(\sqrt{x} - \text{arc tan } \sqrt{x}) + C.$
8. $\frac{3}{8}(x + 1)^{\frac{8}{3}} - \frac{6}{5}(x + 1)^{\frac{5}{3}} + \frac{3}{2}(x + 1)^{\frac{2}{3}} + C.$
9. $\frac{1}{45}(2x^3 + 4)^{\frac{3}{2}}(3x^3 - 4) + C.$

10. $\frac{1}{5}(x^2 + 6)^{\frac{3}{2}}(x^2 - 4) + C.$

12. $\log (\tan \frac{1}{2}x - 2) + C.$

13. $\frac{2}{\sqrt{3}} \text{ arc tan } \left(\frac{2 \tan \frac{1}{2}x - 1}{\sqrt{3}}\right) + C.$

14. $2 \log (\sqrt{x} + \sqrt{x - 4}) - \frac{2\sqrt{x - 4}}{\sqrt{x}} + C.$

15. $-\frac{4}{15} (4 - \sqrt{x})^{\frac{3}{2}} (8 + 3\sqrt{x}) + C.$

16. $e^x - 6 \log (e^x + 2) + C.$

17. $-\frac{1}{r}\sqrt{\frac{2r - x}{x}} + C.$

18. $\frac{1}{r}\sqrt{\frac{x - 2r}{x}} + C.$

Art. 91, pp. 177–178

1. $-e^{-x}(x + 1) + C.$

2. $\frac{xa^x}{\log a} - \frac{a^x}{\log^2 a} + C.$

3. $\frac{e^{2x}}{4}(2x^2 - 2x + 1) + C.$

4. $\frac{x^2a^x}{\log a} - \frac{2xa^x}{\log^2 a} + \frac{2a^x}{\log^3 a} + C.$

5. $x \log x - x + C.$

6. $\frac{x^3 \log x}{3} - \frac{x^3}{9} + C.$

7. $x \log^2 x - 2x \log x + 2x + C.$ 8. $\frac{1}{8} \sin 2t - \frac{1}{4}t \cos 2t + C.$

9. $\frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x + C.$

10. $-\frac{1}{2}\theta^2 \cos 2\theta + \frac{1}{2}\theta \sin 2\theta + \frac{1}{4} \cos 2\theta + C.$

11. $x \operatorname{arc\,sin} x + \sqrt{1 - x^2} + C.$ 12. $x \operatorname{arc\,cos} x - \sqrt{1 - x^2} + C.$

13. $x \operatorname{arc\,tan} x - \frac{1}{2} \log (1 + x^2) + C.$

14. $\frac{1}{2}e^x(\sin x + \cos x) + C.$ 15. $\frac{1}{5}e^x(\sin 2x - 2 \cos 2x) + C.$

16. $\frac{4}{17}e^{2x}(2 \sin \frac{1}{2}x - \frac{1}{2} \cos \frac{1}{2}x) + C.$ 17. $\frac{9}{10}e^{\frac{x}{3}}(\frac{1}{3} \cos x + \sin x) + C.$

18. $\frac{1}{2}x \tan 2x + \frac{1}{4} \log \cos 2x + C.$ 19. $-\frac{1}{3}x \cot 3x + \frac{1}{9} \log \sin 3x + C.$

20. $\frac{1}{8} \cos x \sin 3x - \frac{3}{8} \sin x \cos 3x + C.$

21. $\frac{1}{2}[(x^2 + 1) \operatorname{arc\,tan} x - x] + C.$

22. $\frac{1}{6}[2x^3 \operatorname{arc\,cot} x + x^2 - \log (x^2 + 1)] + C.$

23. $\frac{1}{2} \sec x \tan x + \frac{1}{2} \log (\sec x + \tan x) + C.$

24. $\frac{1}{2} \sec x \tan x - \frac{1}{2} \log (\sec x + \tan x) + C.$

25. $\dfrac{x^2}{4} + \frac{1}{2}(x \sin x + \cos x) + C.$ 26. $\dfrac{x^2}{4} - \frac{1}{8}(2x \sin 2x + \cos 2x) + C.$

27. $2\left[\sin \sqrt{x} - \sqrt{x} \cos \sqrt{x}\right] + C.$ 28. $\dfrac{4x^3}{3} + \frac{1}{2}e^{2x} + 4(xe^x - e^x) + C.$

29. $\frac{1}{3}x^3 + \frac{1}{2}x - \frac{1}{8} \sin 4x - x \cos 2x + \frac{1}{2} \sin 2x + C.$

30. $3x \tan x - 5 \log \cos x + C.$

31. $\frac{1}{4} \sec^3 x \tan x + \frac{3}{8}[\sec x \tan x + \log (\sec x + \tan x)] + C.$

32. $\frac{1}{4} \sec^3 x \tan x - \frac{1}{8}[\sec x \tan x + \log (\sec x + \tan x)] + C.$

Art. 92, pp. 182–183

1. $x + 3 \log (x - 3) + C.$ 2. $2x + 7 \log (x - 1) + C.$

3. $-\frac{2}{3}x + \frac{11}{9} \log (3x - 5) + C.$

4. $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 2 \log (x - 1) + C.$

5. $\frac{3}{2} \log (x - 2) - \frac{1}{2} \log (x + 4) + C.$

6. $5 \log (x + 3) - 2 \log (x + 2) + C.$

7. $x + \frac{31}{3} \log (x - 5) - \frac{7}{3} \log (x - 2) + C.$

8. $6 \log x - 5 \log (x - 1) + C.$

9. $3 \log x + 4 \log (x - 1) - \log (x - 3) + C.$

10. $\frac{1}{2} \log x - \log (x - 1) + \frac{1}{2} \log (x - 2) + C.$

11. $4 \log (x + 2) - 2 \log (x - 3) - \log (x - 1) + C.$

12. $\frac{11}{27} \log (x - 4) - \frac{5}{27} \log (2x + 1) - \frac{2}{9} \log (x - 1) + C.$

13. $x + \frac{1}{3} \log (x + 1) + \frac{2}{3} \log (x - 2) - 2 \log (x + 2) + C.$

14. $-\frac{1}{6} \log x + \frac{3}{2} \log (x - 1) - \frac{11}{2} \log (x - 2) + \frac{31}{6} \log (x - 3) + C.$

18. $\log (2e^x - 1) - x + C.$

19. $-\dfrac{1}{x - 1} + 2 \log (x - 1) - 4 \log (x - 3) + C.$

20. $-\dfrac{1}{4(x + 2)} - \frac{11}{32} \log (x + 2) + \frac{11}{32} \log (x - 6) + C.$

21. $\frac{1}{16}\left[\log (x - 2) - \log (x + 2) + \dfrac{4}{x}\right] + C.$

22. $\frac{1}{2}x^2 + 2x + \frac{1}{4}\left[\log x + 15 \log (x - 2) - \dfrac{2}{x} - \dfrac{2}{x^2}\right] + C.$

23. $\log (x - 4) - \log (x - 3) + \dfrac{1}{x - 3} + C.$

24. $\frac{5}{2}\log(x-1) - \frac{11}{2}\log(x-3) - \dfrac{6}{x-1} + C.$

27. $3\log(x-4) - \log(x^2+2x+3) - \dfrac{3}{\sqrt{2}}\arctan\dfrac{x+1}{\sqrt{2}} + C.$

28. $\frac{3}{2}\log(x^2+x+1) - \log x + \dfrac{11}{\sqrt{3}}\arctan\dfrac{2x+1}{\sqrt{3}} + C.$

29. $\log(x-1) + 2\log(x-4) + \frac{3}{2}\log(x^2+x+4)$
$$- \frac{\sqrt{15}}{3}\arctan\frac{2x+1}{\sqrt{15}} + C.$$

30. $\dfrac{7}{\sqrt{5}}\arctan\dfrac{x}{\sqrt{5}} - \dfrac{3}{x} - \log x + C.$

CHAPTER XVIII

Art. 95, pp. 187–188

2. $6\frac{7}{8}$; $11\frac{3}{8}$. **3.** 49. **4.** 64. **5.** 64.

6. $\begin{cases} \left(\dfrac{4}{n}\right)^5[1^4+2^4+3^4+\cdots+(n-1)^4]; \\ A = \frac{1024}{5}. \end{cases}$

Art. 96, pp. 190–191

3. 24. **4.** 5 log 5. **5.** $\frac{32}{3}$. **6.** $\frac{100}{3}$.

7. $\dfrac{\pi}{4}$. **8.** 36. **9.** 4. **10.** 36.

11. 1. **12.** π. **13.** $\frac{128}{3}$. **14.** $\frac{2}{3}$.

15. $\frac{64}{3}$. **16.** 4 log 2. **17.** $\frac{32}{3}$. **18.** 3.

19. $\frac{32}{3}$. **20.** $2\pi - \frac{4}{3}$. **21.** $\frac{125}{6}$. **22.** $\frac{63}{4}$; $\frac{16}{3}$.

Art. 101, pp. 198–199

5. $\frac{32}{5}$. **6.** $\frac{2}{3}$. **7.** $\dfrac{\pi}{4}$. **8.** $\dfrac{e^2+1}{4}$.

9. $\frac{1}{2}\pi - 1$. **10.** $\frac{48}{5}$. **11.** $\dfrac{e-1}{2e}$. **12.** 18.

13. $\frac{5}{24}$. **14.** $\dfrac{\pi}{4}$. **15.** $\log\sqrt{3}$. **16.** $\dfrac{\pi}{4} - \frac{2}{3}$.

17. $-\frac{1}{2}$. **18.** $\frac{2}{3}$. **19.** π. **20.** $\dfrac{3\sqrt{3}-\pi}{3}$.

21. $2 + 2\log\frac{5}{3}$. **22.** $\frac{20}{3}$. **23.** $\frac{128}{5}$. **24.** $\dfrac{\pi a^2}{4}$.

25. 5. **26.** $\dfrac{2-\sqrt{2}}{3}$. **28.** πab. **29.** $3\pi a^2$.

30. $\dfrac{3\pi a^2}{8}$. **31.** $\frac{125}{6}$. **32.** $32\frac{1}{4}$.

Art. 102, pp. 202–203

1. No value. **2.** 1. **3.** $\frac{1}{2}$. **4.** No value.

5. 1. **6.** -1. **7.** $\frac{1}{2}\pi$. **8.** log 2.

9. 2. **10.** No value. **11.** No value. **12.** 0.

13. No value **14.** $\frac{9}{2}$. **15.** $\frac{1}{2}\pi$. **16.** $\log(2+\sqrt{3})$.

17. $\frac{\pi a^2}{4}$. **18.** $\frac{\pi}{12}$. **19.** No value. **20.** $\frac{1}{3}\pi$.

21. $4\pi a^2$. **22.** 12π.

CHAPTER XIX

Art. 105, pp. 207–209

7. 16π. **8.** $\frac{3}{2}\pi a^2$. **9.** $\frac{1}{2}\pi a^2$. **10.** 4π.

11. $\frac{\pi a^2}{4}$. **12.** a^2. **13.** a^2. **14.** $\frac{19\pi}{2}$.

15. $2\sqrt{3} - \frac{2\pi}{3}$. **16.** 3π. **17.** $\frac{2\pi - 3\sqrt{3}}{2}$. **18.** $2a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$.

19. $\frac{5\pi}{4}$. **20.** $4\left(\sqrt{3} - \frac{\pi}{3}\right)$. **21.** $\frac{4}{3}$. **23.** 36.

Art. 107, pp. 211–213

3. $\frac{4\pi r^3}{3}$. **4.** $\frac{\pi r^2 h}{3}$. **5.** 48π cu. in.

6. $\frac{1}{2}\pi^2$. **7.** $\frac{4}{3}\pi ab^2$; $\frac{4}{3}\pi a^2 b$. **8.** $8\pi \log 2$.

9. $\pi - \frac{\pi^2}{4}$. **10.** $\frac{128\pi}{3}$. **11.** $\pi^2 - 2\pi$. **12.** $\frac{1}{2}\pi$.

13. 2π. **14.** $2\pi^2 - 3\sqrt{3}\pi$; $\frac{2}{3}\pi^2 + \sqrt{3}\pi$.

15. $\frac{1}{2}\pi(e^2+1)$. **16.** $48\pi\sqrt{3} - \frac{64\pi^2}{3}$. **17.** $5\pi^2 a^3$.

18. $6\pi^3 a^3$. **19.** $4\pi^2$. **20.** $\frac{\pi r^3}{3}(2-\sqrt{2})$.

21. $\frac{1}{2}\pi a^3$. **22.** $\frac{2\pi}{3}(6^{\frac{3}{2}} - 11)$. **23.** $\frac{4\pi}{3}[16^3 - (8\sqrt{3})^3]$.

24. $2\pi^2 r^3$. **25.** $2\pi^2 r^2 R$.

Art. 108, pp. 215–216

1. $A(z) = \frac{3}{8}(16-z)^2$. $v = 512$ cu. in. **2.** $\frac{abc}{6}$. **3.** $\frac{1024}{3}$.

4. 36π. **5.** π. **6.** 3π. **7.** $\frac{16r^3}{3}$.

8. $\frac{4}{3}\pi abc$. **9.** 18 cu. ft. **10.** πa^3. **12.** 4π.

Art. 110, pp. 220–222

3. $\frac{13}{3}$. **4.** $\frac{a}{2}\left(e^{\frac{x_1}{a}} - e^{-\frac{x_1}{a}}\right)$. **5.** $6a$. **7.** $\frac{\pi^2 a}{8}$.

9. 32. **12.** $\frac{12\pi a^2}{5}$. **13.** $\frac{\pi}{6}(17^{\frac{3}{2}} - 1)$. **14.** 49π sq. in.

15. $\pi r\sqrt{r^2 + h^2}$. 16. $4\pi^2 Rr$. 17. $\dfrac{64\pi a^2}{3}$.

18. $2\pi[\sqrt{2} + \log(\sqrt{2} + 1)]$. 19. $\pi[\sqrt{2} + \log(\sqrt{2} + 1)]$.

20. $4\pi^2 r^2$.

CHAPTER XX

Art. 114, pp. 226–227

4. $\dfrac{2}{\pi}$. 5. $\frac{1}{3}$. 6. $\frac{20}{3}$. 7. $\dfrac{\pi r}{4}$.

9. $\frac{50}{3}$. 11. 27. 12. $-\frac{68}{3}$. 14. $\dfrac{16r^3}{3}$.

15. 48π cu. in. 16. 44 m.p.h.; 45 m.p.h.

Art. 116, pp. 230–231

1. $A_t = 244$; $A_s = A = 242\frac{2}{3}$.
3. $A_t = 8.41$; $A_s = 8.78$; $A = 9$.
5. $A_t = 175$; $A_s = A = 180$.
7. $A_t = 16.10$; $A_s = 16.07$.
9. 34.12.
11. $A_t = 1.46$; $A_s = 1.49$.
13. 1.17.
15. 66.7.

CHAPTER XXI

Art. 117, p. 234

2. $\bar{x} = 0$; $\bar{y} = \dfrac{2r}{\pi}$.
4. $\bar{x} = \dfrac{18}{\pi}$ in.; $\bar{y} = 0$.
5. $\bar{x} = \bar{y} = \dfrac{2a}{5}$.
7. $\bar{x} = \pi a$; $\bar{y} = \dfrac{4a}{3}$.
9. $4\pi^2 rR$.
10. πrs; $\begin{cases} r = \text{radius} \\ s = \text{slant height.} \end{cases}$
11. $\pi(r + R)s$ $\begin{cases} r \text{ and } R = \text{radii of bases} \\ s = \text{slant height.} \end{cases}$

Art. 119, pp. 239–240

1. $\bar{x} = \frac{12}{5}$; $\bar{y} = 0$. 2. $\bar{x} = \bar{y} = \frac{18}{5}$. 3. $\bar{x} = 3$; $\bar{y} = -\frac{8}{5}$.

4. $\bar{x} = \dfrac{\pi - 2}{2}$; $\bar{y} = \dfrac{\pi}{8}$. 5. $\bar{x} = \dfrac{2h}{3}$; $\bar{y} = \dfrac{r}{3}$. 6. $\bar{x} = 4$; $\bar{y} = 5$.

7. $\bar{x} = \dfrac{5a}{6}$; $\bar{y} = 0$. 8. $\bar{x} = \bar{y} = \dfrac{a}{5}$. 9. $\bar{x} = \dfrac{4a}{3\pi}$; $\bar{y} = \dfrac{4b}{3\pi}$.

10. $\dfrac{2\sqrt{15}}{3}$ in. from 4-in. side. 11. $\bar{x} = \pi a$; $\bar{y} = \dfrac{5a}{6}$.

12. $\bar{x} = \frac{16}{15}$; $\bar{y} = \frac{128}{105}$. 17. $2\pi^2 r^2 R$. 19. $\dfrac{\pi h}{3}(r^2 + R^2 + rR)$.

Art. 120, pp. 242–243

2. (0, 0, 3 ft.). 3. (3 in., 3 in., 4 in.). 4. $\left(\dfrac{a}{4}, \dfrac{b}{4}, \dfrac{c}{4}\right)$.

5. $\frac{1}{4}h$ above base. 6. $\frac{2}{3}h$ from end. 7. $\frac{2}{5}h$ from base.
8. $\frac{3}{8}r$ from plane surface. 9. $\frac{8}{15}r$ from plane surface.
10. (4, 0, 0). 11. $(0, 0, \frac{40}{3})$. 12. 6 in. above base.
13. $6\frac{7}{8}$ in. from end. 14. 8.06 ft. below top. 15. $\bar{y} = \dfrac{4r}{3\pi}$.
16. $\bar{x} = \dfrac{3a}{8}$. 17. $\dfrac{h}{3}$ above base. 18. $\frac{3}{16}\pi r$, $\frac{3}{32}\pi h$.

CHAPTER XXII

Art. 124, pp. 248–249

1. $\dfrac{1{,}024}{15}$; $\dfrac{1{,}024}{7}$. 2. $\dfrac{\pi ab^3}{4}$; $\dfrac{\pi a^3 b}{4}$. 3. $\frac{2}{9}$; $\dfrac{\pi^2 - 8}{4}$.
4. $\frac{512}{15}$; $\frac{32}{3}$. 5. $\frac{4}{9}$. 6. $\frac{256}{5}$; $\sqrt{\dfrac{24}{5}}$. 7. $16 - 4\pi$.
8. $\dfrac{e^3 - 1}{9}$; $e - 2$. 11. $\dfrac{bh^3}{36}$. 12. $\dfrac{a^4}{12}$.
17. 1,536 in.^{4}. 20. 301; 16.

Art. 130, pp. 254–256

2. $\frac{2}{5}Mr^2$. 3. $\frac{3}{2}Mr^2$. 4. $\frac{2}{3}Ma^2$. 5. $\frac{1}{3}ML^2$.
6. $\frac{1}{10}Ma^2$. 7. $\dfrac{3M}{20}(r^2 + 4h^2)$. 8. $\frac{2}{5}Mb^2$.
9. $\dfrac{512\pi}{3}$. 10. $\dfrac{3\pi^2}{16}$. 11. $\frac{1}{2}\pi^2 r^2 R(4R^2 + 3r^2)$.
14. $\frac{7}{5}Mr^2$. 17. 1:2. 19. $\dfrac{2L^4}{3}$.
20. $\dfrac{bh^3}{12} + \dfrac{hb^3}{12}$. 21. $\dfrac{671\pi}{2}$.

CHAPTER XXIII

Art. 134, pp. 261–263

3. $30w$ lb.; $\frac{1}{15}$ ft. below centroid. 4. $16w$ lb.
5. 6,750 lb.; 1 ft. above bottom. 6. top $562\frac{1}{2}$ lb.; bottom $2{,}812\frac{1}{2}$ lb
7. 267 lb. 8. $20\pi w$ lb.
9. $100(3\pi - 4)$ lb.; $100(3\pi + 4)$ lb.
10. $60\pi w$. 11. $24w\sqrt{2}$ lb.; $\dfrac{8w}{3}$ lb.
12. $800w$ lb. $y' = \frac{20}{3}$ ft. 13. 3.14 ft. below surface; $27w$ lb.

Art. 136, pp. 266–267

2. 1,152 in.-lb. 3. 160 in.-lb. 5. 72 ft.-lb. 6. 8 lb.
9. (144)(6,000) log $\frac{40}{3}$ ft.-lb. 10. $\dfrac{9{,}225\pi}{2}$ ft.-lb.
12. $240\pi w$ ft.-lb. 13. $\dfrac{2{,}752\pi w}{3}$ ft.-lb.

CHAPTER XXIV

Art. 138, pp. 270–272

4. 2. **5.** $-\frac{1}{2}$. **6.** $-\frac{1}{3}$; $-\frac{2}{3}$. **7.** -1.

9. $3(x^2 + y^2)$; $6y(x - y)$. **11.** $\dfrac{y^2}{(y - x)^2}$; $\dfrac{y^2 - 2xy}{(y - x)^2}$.

13. $-\dfrac{y}{x\sqrt{x^2 - y^2}}$; $\dfrac{1}{\sqrt{x^2 - y^2}}$. **15.** $\frac{1}{2}y \sin \theta$; $\frac{1}{2}x \sin \theta$; $\frac{1}{2}xy \cos \theta$.

17. $z = 4x + 2$; $y = 2$. **18.** $x + 2z = 5$; $y = 2$.

Art. 139, pp. 274–275

1. $x + y - z = 2$; $\begin{cases} x - y = 0 \\ y + z = 4. \end{cases}$ **2.** $4x + 2y - z = 7$; $\begin{cases} x - 2y = 0 \\ y + 2z = 7. \end{cases}$

3. $5x + 2y + 3z = 38$; $\begin{cases} 2x - 5y = 0 \\ 3y - 2z = 0. \end{cases}$

4. $x + 2y + 2z = 12$; $\begin{cases} 2x - y = 6 \\ y - z = 0. \end{cases}$

6. $2y + 2z - x = 8$; $\begin{cases} 2x + y = -2. \\ y - z = 1. \end{cases}$

Art. 140, pp. 277–279

1. 75.7. **2.** 210.8. **4.** 4.015.

6. $6(x + y)dx + (6x + 4)dy$. **8.** $\dfrac{1}{1 - y^2}dx + \dfrac{2xy}{(1 - y^2)^2}dy$.

10. $\frac{1}{2}y \sin \theta\, dx + \frac{1}{2}x \sin \theta\, dy + \frac{1}{2}xy \cos \theta\, d\theta$.

12. 8.8 cu. ft. **14.** 0.035. **16.** 42.6 sq. ft.

17. $\frac{1}{3}$ ft. per second. **19.** 5 per cent. **22.** 1 per cent.

Art. 142, pp. 283–284

1. 0.192 cu. in. per minute. **2.** 0.0866 radian per minute.

3. $\dfrac{4 - \pi}{3\pi}$ ft. per minute. **4.** -9 units per minute; $\alpha = 45°$.

5. $\dfrac{dz}{dx} = -\dfrac{5}{2}$; $\dfrac{dz}{ds} = -\dfrac{\sqrt{5}}{2}$. **6.** $-\dfrac{12\sqrt{5}}{5}$.

9. $\dfrac{3\sqrt{5}}{5}$. **10.** $x = \frac{26}{5}$; $y = \frac{17}{5}$. **11.** $-\dfrac{\sqrt{2}}{4}$.

12. $-2\sqrt{13}$. **14.** $2\pi r(h + r^2)$.

Art. 144, pp. 286–287

7. $\dfrac{x}{4z}$; $\dfrac{y}{4z}$. **8.** $-\dfrac{c^2x}{a^2z}$; $-\dfrac{c^2y}{b^2z}$. **10.** $\dfrac{yz - x^2}{z^2 - xy}$; $\dfrac{xz - y^2}{z^2 - xy}$.

12. $2(x + 4y)$. **14.** $\dfrac{2xy}{(x + y)^3}$. **16.** $12xy + 24y^2$.

CHAPTER XXV

Art. 146, p. 290

1. 6.　2. -144.　3. π.　4. $\frac{\pi r^2}{4}$.
5. $\frac{1}{6}$.　6. $\frac{2}{3}$.　7. 1.　8. π^2.
9. $\frac{17}{144}$.　10. $\frac{11}{80}$.　11. $\frac{4\pi r^3}{3}$.　12. $\frac{4\pi r^3}{3}$.

Art. 148, pp. 295–296

2. πab.　3. $\frac{1}{10}$.　4. 36.　5. $\frac{16}{3}$.
6. $\frac{32}{3}$.　7. $\sqrt{2} - 1$.　8. $\frac{448}{15}$.　9. 16.
10. 4π.　11. $\frac{16r^3}{3}$.

Art. 151, p. 299

1. $\frac{\pi r^4}{4}$.　3. $\frac{r^4}{8}$.　4. 576 in.^{4}.
5. $\frac{\pi r^4}{2}$.　6. $\frac{16}{3}$.　8. $\frac{4r^3}{3}$.
11. $\frac{1}{2}Mr^2$.　12. $M\left(\frac{r^2}{4} + \frac{h^2}{3}\right)$.　16. 288 cu. in.

Art. 153, pp. 303–304

2. a^2.　3. $\frac{3\pi}{2}$.　4. $\pi - \frac{3\sqrt{3}}{2}$.　5. $\frac{a^2\pi}{4}$.
6. $\frac{\pi}{2}$.　7. $\frac{16\pi}{3} + 8\sqrt{3}$.　8. $4\left(\sqrt{3} - \frac{\pi}{3}\right)$.
9. $\frac{5\pi}{4}$.　12. 8π.　13. 16π.　14. $\frac{16\pi}{3}$.
15. πr^3.　16. $\frac{244\pi}{3}$.　17. $\frac{128}{9}(3\pi - 4)$.
18. $\frac{256}{9}$.　19. 24π.　20. $\frac{16\pi}{3}$.

CHAPTER XXVI

Art. 159, p. 313

3. Divergent.　5. Convergent.　7. Divergent.
9. Convergent.　11. Divergent.　13. Convergent.

Art. 162, pp. 317–318

3. Divergent.　5. Convergent.　7. Divergent.　9. Convergent.
11. Convergent.　13. Divergent.　15. Convergent.　17. Convergent.
19. Convergent.　21. Convergent.

Art. 164, pp. 320–321

1. Conditional
3. Conditional.
5. Absolute.
7. Conditional.
9. Conditional.
11. Absolute.
13. 3.0366.
14. About 400 terms.
15. 0.2874.

Art. 166, pp. 324–325

1. $-1 < x < +1$.
2. $-1 < x < +1$.
3. $-2 < x < +2$.
4. All values.
5. $-5 \leqq x < +5$.
6. $-2 \leqq x \leqq +2$.
7. All values.
8. $-1 < x < +1$.
9. $-3 < x < +3$.
10. $x = 0$.
11. $2 < x < 4$.
12. All values.
13. $-2 < x \leqq 0$.
14. $3 \leqq x \leqq 5$.
15. $1 \leqq x \leqq 3$.
16. $1 \leqq x \leqq 3$.

CHAPTER XXVII

Art. 168, pp. 329–330

6. $1 + (\log a)x + \dfrac{(\log a)^2}{2!}x^2 + \dfrac{(\log a)^3}{3!}x^3 + \cdots$. All values.

7. $-\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots\right)$. $-1 \leqq x < +1$.

8. $1 - x + x^2 - x^3 + \cdots$. $-1 < x < +1$.

9. $-\dfrac{\sqrt{2}}{2}\left(1 - x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots\right)$. All values.

12. $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots$.

13. $1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + \cdots$.

14. $x + \dfrac{x^3}{6} + \dfrac{x^5}{24} + \dfrac{61x^7}{5{,}040} + \cdots$.

15. $-\left(\dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45} + \cdots\right)$.

20. $\theta < 0.144$ radian.

Art. 170, p. 334

1. $\frac{1}{2}\left[1 + \sqrt{3}\left(x - \dfrac{\pi}{6}\right) - \dfrac{\left(x - \frac{\pi}{6}\right)^2}{2!} - \dfrac{\sqrt{3}\left(x - \frac{\pi}{6}\right)^3}{3!} + \cdots\right]$.

2. $\dfrac{\sqrt{2}}{2}\left[1 - \left(x - \dfrac{\pi}{4}\right) - \dfrac{\left(x - \frac{\pi}{4}\right)^2}{2!} + \dfrac{\left(x - \frac{\pi}{4}\right)^3}{3!} + \cdots\right]$.

3. $1 + \frac{1}{2}(x - 1) - \dfrac{1}{2^2 2!}(x - 1)^2 + \dfrac{3}{2^3 3!}(x - 1)^3 - \dfrac{3 \cdot 5}{2^4 4!}(x - 1)^4 + \cdots$.

4. $e\left[x + \dfrac{(x - 1)^2}{2!} + \dfrac{(x - 1)^3}{3!} + \cdots\right]$.

5. $\log 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3 - \cdots$.

7. $1 + 2x + 3x^2 + 4x^3 + \cdots$.

8. $2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right)$.

9. $x + \dfrac{x^3}{2 \cdot 3} + \dfrac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \dfrac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$.

10. 0.856.

11. $x + x^2 + \frac{2x^3}{3!} - \frac{4x^5}{5!} - \cdots$.

12. $-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots$.

Art. 173, pp. 340–341

2. $x = \sqrt{7}$.

3. $x = \sqrt{3} + 1$.

7. 1.6487.

CHAPTER XXVIII

Art. 178, pp. 349–350

3. $y = \sin x + 2 \cos x$.

5. $y^2 = 4x + C$.

7. $xy = C$.

8. $x + y = Cxy$.

9. $\log (x + 1) = C - \frac{1}{y}$.

10. $\log \frac{1 + y}{1 - y} = 2x + C$.

11. $\log (xy) = x - y + C$.

12. $1 + y^2 = C(1 + x^2)$.

13. $\log (E - Ri) = -\frac{Rt}{L} + CR$.

14. $\sqrt{1 - y^2} = \frac{1}{x} + C$.

15. $\log (x^2 y) = y + C$.

Art. 179, p. 352

1. $x^2 + 2xy = C$.

2. $x^2 + y^2 = Cx^3$.

3. $x^2(x^2 + 2y^2) = C$.

4. $\log Cy = \frac{y}{x}$.

5. $y^2 + xy = Cx$.

6. $\frac{y^3}{x^3} + 3 \log x = C$.

7. $x^2 + y^2 - 4xy = C$.

8. $\log Cy + 2\sqrt{\frac{x}{y}} = 0$.

9. $y = x \log x$.

10. $y^2 + 2xy - x^2 = 2$.

12. $y^2 + 2xy - x^2 - 4y + 8x = C$.

Art. 181, pp. 355–356

1. $x^2 + xy - y = C$.

2. $x^3 + 2xy + y^2 + y = C$.

3. $y \sin x + x = C$.

4. $x^2 y + \log x = C$.

5. $x^3 y + \log y = C$.

6. $xy(1 - x^2) = C$.

7. $2y^2 - x^2 - 2xy + 4y + 2x = C$.

8. $3xy^2 - 3x^2 y + 2x^3 = C$.

9. $x^3 + 3xy^2 = C$.

10. $x^2 + y^2 = Cx$.

11. $x^2 + 2xy - 2y^2 = C$.

12. $y^2 - \log (2x + 8) = C$.

14. $y = Cx$.

15. $\frac{y}{x} - \log x = C$.

16. $\frac{x}{y} + \frac{x^2}{2} = C$.

17. $\frac{y}{x^2} = \frac{x^2}{2} + C$.

18. $y^2(x + 1) = C$.

Art. 183, pp. 358–359

1. $4y - 2x + 1 = Ce^{-2x}$.

2. $ye^x = x + C$.

3. $x + y + 1 = Ce^x$.

4. $xy = \frac{x^3}{3} + C$.

5. $2x + 2y + 1 = Cx^2$.

6. $xy + \cos x = C$.

7. $xy + x \cos x = \sin x + C$.

8. $2ye^x = e^x(\sin x - \cos x) + C$.

9. $(y - x^2 - 1)^2 = C(x^2 + 1)$. **10.** $\frac{y}{x^2} + \frac{1}{3x^3} = C$.
11. $2y \log x - \log^2 x = C$. **12.** $x^2y = e^x + C$.
13. $x^2y - 3x = C$. **14.** $2xye^x - e^{2x} = C$.
16. $\frac{1}{x^2y} + \log x = C$. **17.** $x^2(y^3 - 1) = C$.
18. $y^{-2}e^x + 2x = C$. **19.** $x + y + 1 = e^x$. **20.** $2y + 1 = 3e^{x^2}$.

Art. 185, pp. 362–363

1. $y = Ae^{2x} + Be^x$. **2.** $y = Ae^{4x} + Be^{-3x}$.
3. $y = Ae^{-\frac{1}{2}x} + Be^{3x}$. **4.** $y = Ae^{2x} + Be^{-2x}$.
5. $y = A + Be^{2x}$. **6.** $y = Ae^x + Be^{2x} + Ce^{3x}$.
7. $y = Ae^x + Be^{2x} + Ce^{-2x}$. **8.** $y = Ae^{2x} + Be^{\frac{1}{2}x} + Ce^{-\frac{1}{2}x}$.
9. $y = Ae^{2x} + Bxe^{2x}$. **10.** $y = Ae^{-x} + Bxe^{-x}$.
11. $y = Ae^{\frac{1}{2}x} + Bxe^{\frac{1}{2}x}$. **12.** $y = e^x(A \cos 2x + B \sin 2x)$.
13. $y = e^{5x}(A \cos x + B \sin x)$. **14.** $y = e^{3x}(A \cos 2x + B \sin 2x)$.
15. $y = Ae^{2x} + e^x(B \cos x + C \sin x)$.
16. $y = Ae^x + Be^{2x} + Cxe^{2x}$.
17. $y = Ae^{2x} + Be^{-2x} + C \cos 2x + D \sin 2x$.
18. $y = e^x - e^{-2x}$. **19.** $y = e^x + e^{-x}$. **20.** $y = xe^x$.
21. $y = e^x \cos 2x$. **22.** $Q = 5e^{-2t}(\cos t + 2 \sin t)$.
23. $x = 2 \cos 2t;\ \frac{dx}{dt} = -4 \sin 2t$. **24.** $y = \frac{1}{2}(e^{\frac{1}{2}\sqrt{g}t} + e^{-\frac{1}{2}\sqrt{g}t})$.
25. $\theta = \theta_1 \cos \sqrt{\frac{g}{L}}t;\ T = 2\pi\sqrt{\frac{L}{g}}$.

Art. 186, pp. 365–366

1. $y = ax^2 + bx + c$;
$y = a \sin x + b \cos x$;
$y = ae^{-x}$.
2. $y = Ae^{2x} + Be^x + \frac{1}{5} \cos x - \frac{3}{5} \sin x$.
3. $y = Ae^{2x} + Be^x + 2x^3 + 9x^2 + 21x + \frac{45}{2}$.
4. $y = A + Be^{2x} - x^3 - 2x^2 - \frac{7}{2}x$. **5.** $y = Ae^{2x} + Be^{4x} + \frac{1}{4}$.
7. $y = Ae^x + Be^{-2x} - 3 \cos x - \sin x$.
9. $y = Ae^{2x} + Be^{3x} + 2e^x$.
11. $y = Ae^{-x} + Be^{\frac{1}{2}x} - 3x^2 + 2x - \frac{14}{3}$.
13. $y = e^{2x}(A \cos 2x + B \sin 2x) + \frac{1}{4}x + 1$.
15. $y = Ae^{-3x} + Be^{2x} + 2xe^{2x}$.
17. $y = A \cos x + B \sin x - x \cos x$.
19. $y = A + Bx + Ce^{-x} + \cos x - \sin x$.
21. $y = A + Be^{-x} + \frac{1}{3}x^3 + 4x$. **23.** $y = 2e^x - x$.